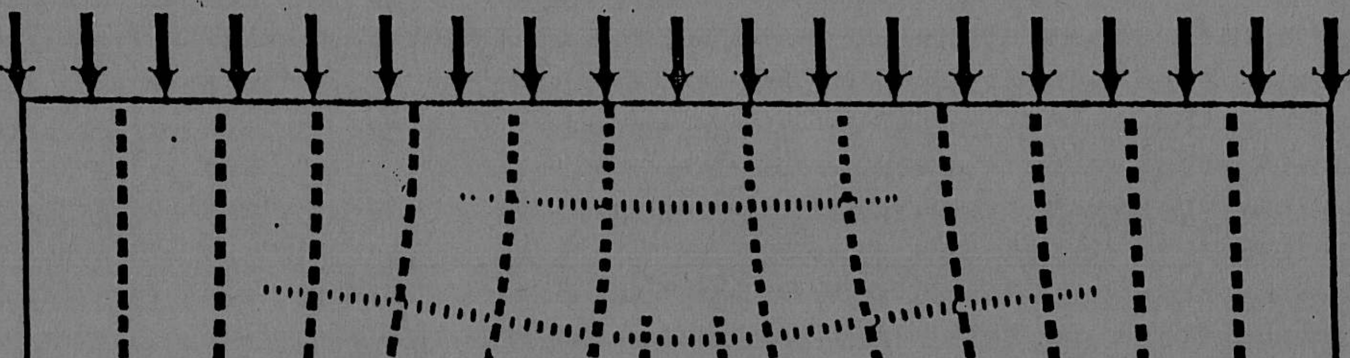
THE CONCEPT OF:

THE FLOW OF FORCES

Excerpt from Notes on Structural Behavior for Architecture students

Professor Waclaw Zalewski



M.I.T. 1975-1980

F L O W O F F O R C E S

"ONCE SCIENTISTS LIKE VOLTA AND AMPÈRE DISCOVERED HOW TO REPRESENT ELECTRICITY IN TERMS OF THE PRESSURES AND FLOWS OF FLUIDS, THEY COULD TRANSPORT MUCH OF WHAT THEY ALREADY KNEW ABOUT FLUIDS TO THE DOMAIN OF ELECTRICITY"
from "The Society Of Mind" by Marvin Minsky

Flow of forces in structures is an allusive expression summarizing our understanding of how structures function; it makes use of the suggestive visual resemblance between lines of forces transmitting loads through the interiors of structures and the patterns of flow in liquids.

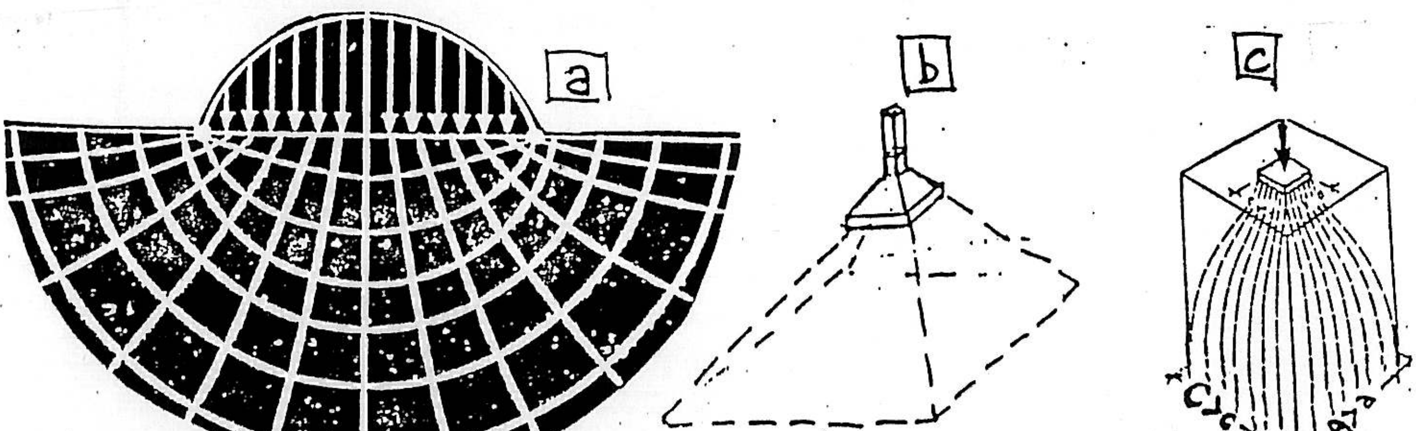
Physically, the interaction of forces in structures has not much in common with the movement of liquids, but the vital fact of the transfer of forces through material is well illustrated by a mnemonic (easy rememberable) analogy with the conduct of water flowing through rivers and canals.

For designers and builders in particular, the intellectual and utilitarian appeal of the metaphoric expression "flow of forces" comes from the visualization of a system of invisible networks of internal forces - too often treated as only algebraic abstraction.

"VISION IS THE ART OF SEEING THINGS INVISIBLE" Jonathan Swift

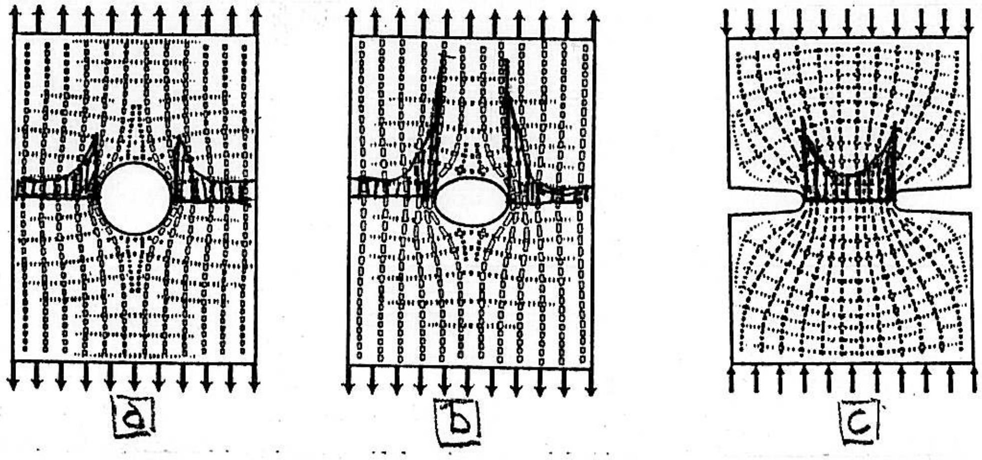
Like a river's flow spreading from its narrow channel into a flat valley, the flow of forces caused by the local pressure exerted on the edge of any physical body exemplifies the characteristic fan-shaped pattern of force penetration into material,(fig.1a). The radiating dispersion of forces is a phenomenon well understood intuitively - conducive to various popular ideas, such as for instance, that loads from foundations of buildings are spreading out through the soil, and are then carried down within a shape similar to a truncated pyramid (cone) with a much larger base,(fig.1b).

Similarly, an easily acceptable geometrical version of the force transfer from the smaller to larger areas is shown in fig.1c.



If one compares the flow of forces in a tensioned or compressed element to the movement of water within boundaries similar to those of the loaded element, the analogy can be easily found between concentrations (enlargement) of forces in the vicinity of holes or other material discontinuities in the stressed element and the local increases of the velocity of water surging around obstacles. Any opening in a stressed element diverts forces from their regular path and intensifies them (often considerably), in analogy to the acceleration of water speeding around obstacles.

The available analytical and experimental evidence relates the degree of stress (internal force related to a unit area) intensification in places around holes to their shape. E.g. when the hole is circular, the magnitudes of stresses are tripled in relation to their values in some distance from the hole (fig.2a). A flattened circle, transformed into elliptical crack with its longer axis perpendicular to the direction of the longitudinal action, can enhance it much more (2b). An additional challenge to the integrity of material is caused by the presence of transversal stresses produced at any turn of the main flow of forces (figs 2a,b,c).

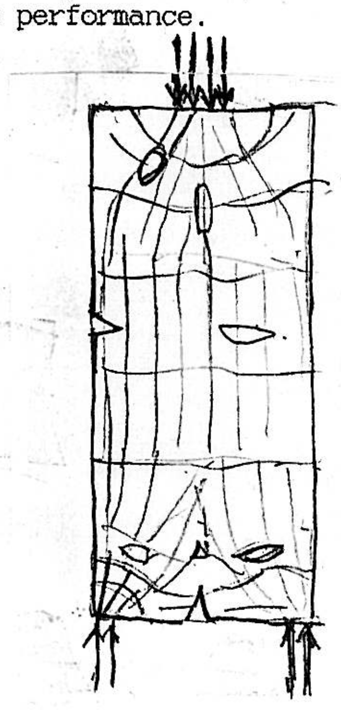


2.

It is practically impossible to avoid occurences of stress magnification in any real structure - in places of local defects in material and also in situations where the flow of forces is either diverted by holes, intrusions and notches or by any abrupt change in the shape of structure. It is important then, to spare no efforts -during the design and construction process - in elimination of everything that stays in the path of the natural, smooth flow of forces.

"THE WAY TO THE PERFECT STRUCTURE IS THROUGH THE KNOWLEDGE OF HOW TO MAKE HOLES IN IT" Robert le Ricolais

Figs.3 and 4 present two cases of perforations in a vertical slab. Whereas the locations and shapes of openings shown in fig 3 would ominously reduce the bearing capacity of a slab, the indentations shown in fig 4 would have only a marginal effect on its performance.



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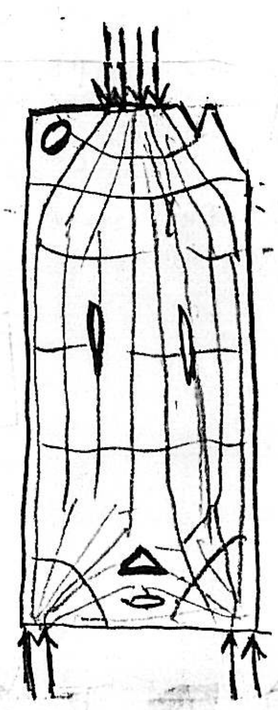


4.

Since the character of some visible phenomena associated with the movement of liquids can be often predicted and evaluated without deeper theoretical studies, it is possible to accept an idea that the similarly subconscious understanding of certain natural events can also include the recognition of typical patterns of force transfer through structures. As a matter of fact, what is needed, in most cases, to approximate the outline of the flow of forces through the structure is the familiarity with certain characteristic rules and geometrical modes governing dispersion of forces in material, combined with an understanding of how such dispersion is affected by the shape of structure.

"THE REMARKABLE, INHERENT SIMPLICITY OF NATURE (EINSTEIN CALLED IT ELEGANCE) ALLOWS THE STRUCTURE TO PERFORM ITS TASK THROUGH TWO ELEMENTARY ACTIONS ONLY: PULLING AND PUSHING. MANY AND VARIED AS THE LOADS MAY BE AND GEOMETRICALLY COMPLICATED AS THE STRUCTURE MAY BE, ITS ELEMENTS NEVER DEVELOP ANY OTHER KIND OF ACTION. THEY ARE EITHER PULLED BY THE LOADS, AND THEN THEY STRETCH, OR ARE PUSHED, AND THEN THEY SHORTEN."

Mario Salvadori

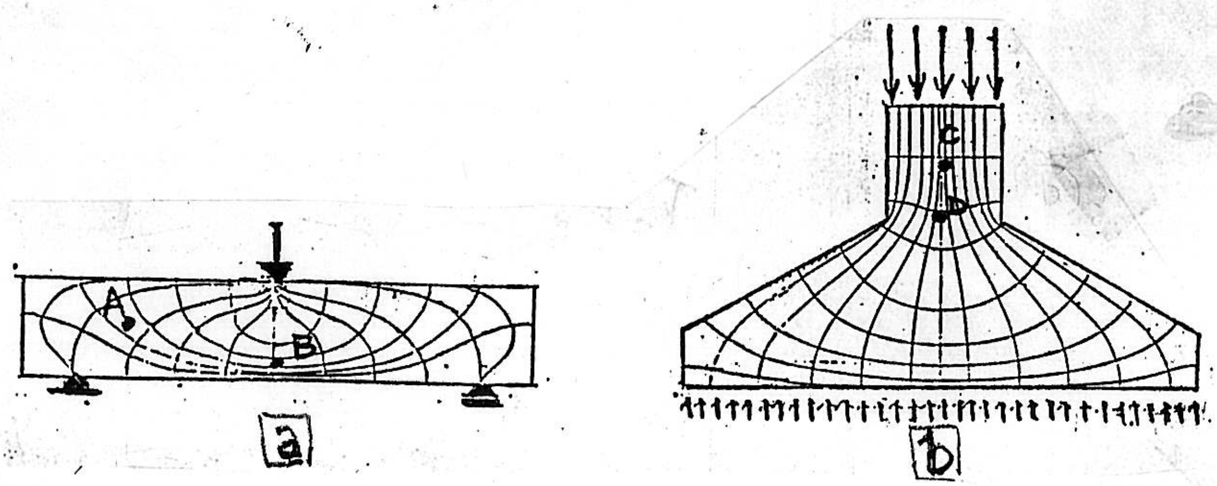
Interpretation of structural behavior in terms of strains and stresses as agents of only two (pushes and pulls) modes of the flow of forces helps to understand the amazing fact that at any point of a body subjected to loads, there are always three mutually perpendicular directions of main actions of internal forces (principal tensions and compressions), which follow lines of the orthogonal system of stress trajectories (paths). Discovery of such an organized manner of force transmission was one of the accomplishments of the Theory of Elasticity which helped to unravel the mysteries of the previous misty imagery of structural behavior, through the establishment of a rational and well-defined foundation for "Resistance of Materials"- the main theoretical discipline of Structural Engineering. It also gave impulse to modifications of the initially purely intuitive concepts of the flow of forces and to the development of a simplified classification of its characteristic patterns.

The acquaintance with the basic geometrical characteristics of flow of forces might allow the designer to get promptly the true meaning of structural action within the body which he is just shaping - it could enable also to foresee the consequences of irrational decisions.

Most arrangements of the flow of forces are more intricate than those shown in figs.1. Take the example of the two well known structural forms:

fig 5a - beam subjected to bending by the central load,

fig 5b - footing of the column carrying a vertical load down to the ground.



5.

Each arrangement of trajectories of principal stresses in the above structures correspond to their particular form and to the type of load action, always satisfying the following two mutually independent requirements:

1. Internal pushes and pulls (usually varying from one point to another) must always equilibrate (in any section) the actions of loads.
2. Material preserves its continuity, in spite of distortions caused by internal forces (compatibility requirement).

Fig.6. illustrates the elementary, multi-directional actions of stresses produced around certain points of both structures (figs.5), in directions coinciding with the course of the flow of forces.

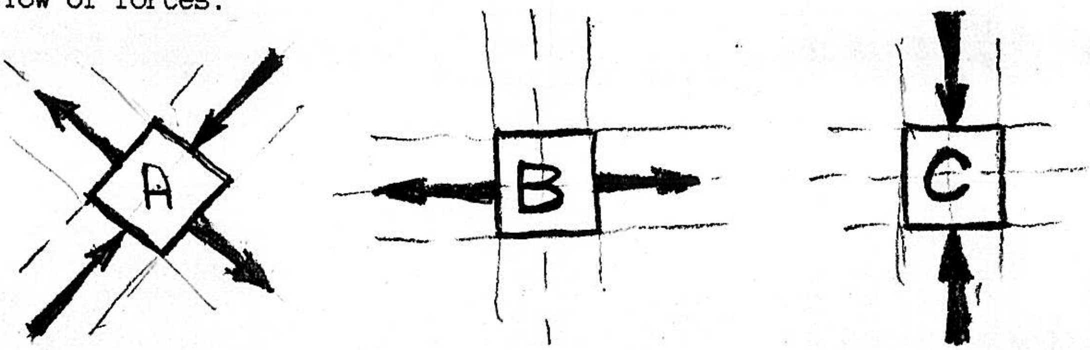
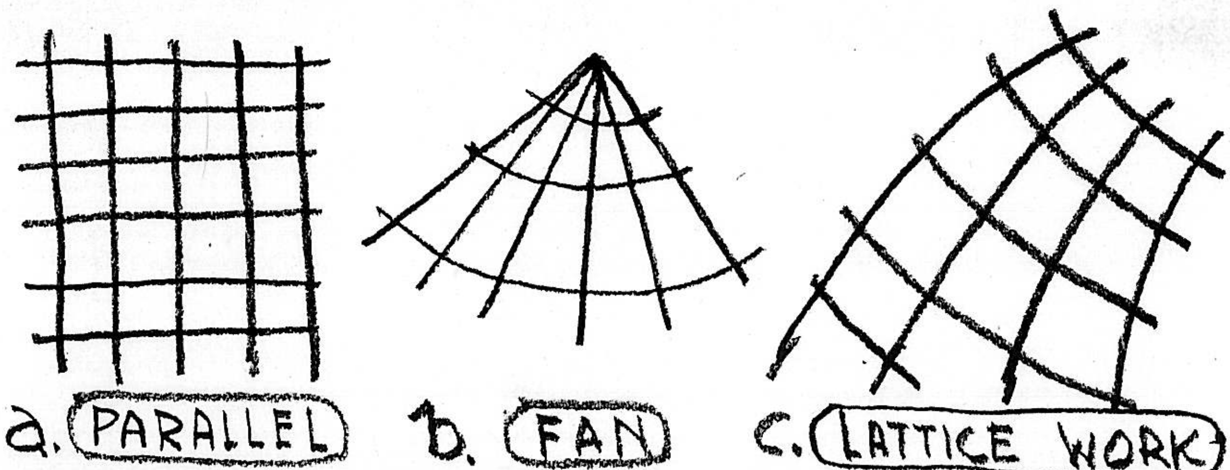


Fig.6.

Just three simplified patterns of the flow of forces represented in figs 7 by the characteristic (rectangular, fan-like and latticed) orthogonal nets and their combinations can provide sufficient (for the purposes of a conceptual design) background of the approximate geometrical arrangements of a majority of modes of force transmission through solid bodies of any shape.



Figs.7

1. A parallel pattern of the flow of forces dominates the upper part of a footing (fig.5b-point C) and exists in all axially loaded structural elements.
2. A fan-like pattern is shown in figs.1; also in the central part of fig.5b -point D, and in fig 11b - it is typically developed under single loads.
3. A lattice-work pattern, typical for beams, is shown in fig.5a - point A.

The interior of any structure affected by loads, can be seen as the material continuum fully packed with particles interacting along lines of principal stresses and transferring tensions and compressions similarly to the action of members of trusses. Yet, there exists an important difference between any real truss with the fixed arrangement of its members (which are the only available conduits for forces) and the invisible, spontaneously developed system of routes of forces in vast interiors of solid bodies. While the response of any truss to loads - in terms of directions and values of forces - is determined by its geometry, the interior of the solid body offers an empty space for an infinite number of configurations of lines of forces. When its material behaves elastically, the structure makes in each case a well defined automatic decision, organizing the transfer of forces in a way making the total elastic energy stored (in structure deformed by loads) always smaller than what would result from any other statically possible manner of force distribution under the same load.

The principle of least work (or of minimum elastic energy) derived rigorously, for linearly elastic materials, in the second half of the XIX century, states that when a structure is in equilibrium under the influence of external actions, the resulting internal forces are such that the work done by them (equal the total energy stored in structure) is minimum.

The same concept of economy and perfection of Nature`s design and work led physiologists of the XIX century to postulate and later to prove that the system of blood circulation (arrangement, forms and sizes of veins and arteries in human and animal bodies) operates with a minimum of effort.

It is interesting to know that long before the discovery of the principle of minimum work, a purely metaphysical concept of the unqualified wisdom of Nature served as the basis of belief in Its inherent efficiency.

Independently of possible theological speculations regarding the implicit wonderfulness of Nature, it is useful and illuminating to identify the seemingly esoteric principle of minimum elastic energy with the tangible requirement of compatibility of strains and deformations, and think about it as of a condition separate and distinct from that of equilibrium of forces.

> "ANOTHER BASIC LAW OF NATURE GOVERNS THE STRUCTURE`S RESPONSE TO LOADS. WITH A JUDICIOUS SENSE OF ECONOMY, OR INTELLIGENT LAZINESS, A STRUCTURE WILL ALWAYS CHOOSE TO CHANNEL ITS LOADS TO THE GROUND BY THE EASIEST OF THE MANY PATHS AVAILABLE. THIS IS THE PATH REQUIRING THE MINIMUM AMOUNT OF WORK ON THE PART OF THE STRUCTURAL MATERIALS AND IS A CONSEQUENCE OF WHAT IS TERMED IN PHYSICS "THE LAW OF LEAST WORK". STRUCTURE BEHAVES HUMANLY IN THIS RESPECT TOO" Mario Salvadori

Work is a concept which, in physics, does not have quite the same meaning as in popular usage, where it seems to imply physical exertion.

The work done by forces increases the energy stored in a structure - both expressions "work" and "energy" being practically synonymous and technically defined as the product of multiplication of the displacement by the component of force, in the direction of displacement. When deformations are proportional to forces (it means within the elastic range) the work done by loads is recoverable; on removal of loading, material returns to its initial state due to the total release of stored energy.

The common example of this is the main spring of a traditional wrist watch. The work done by us in winding the spring is stored in it and later gradually released in overcoming friction in the watch mechanism. If we think of the structure as a elastic body consisting of a large number of particles interconnected by springs, the work done in streching and contracting them would represent the total strain energy in the body.

In the diagram (fig 8), which is typical for an idealized elasto-plastic material behaving similarly in tension and compression, the ascending line O-Y corresponds to the elastic relations between axial force P (either tensile or compressive) and the resulting deformation ΔL. When force is applied gradually (varying from zero to the final value P), the quantity of work W equals the area of the triangle O-A-B in fig.8, [$W = P\cdot\Delta L/2$].

A = area of cross section
f = stress = P/A
ΔL = elongation or shortening
E = modulus of elasticity
V = A·L = volume of element

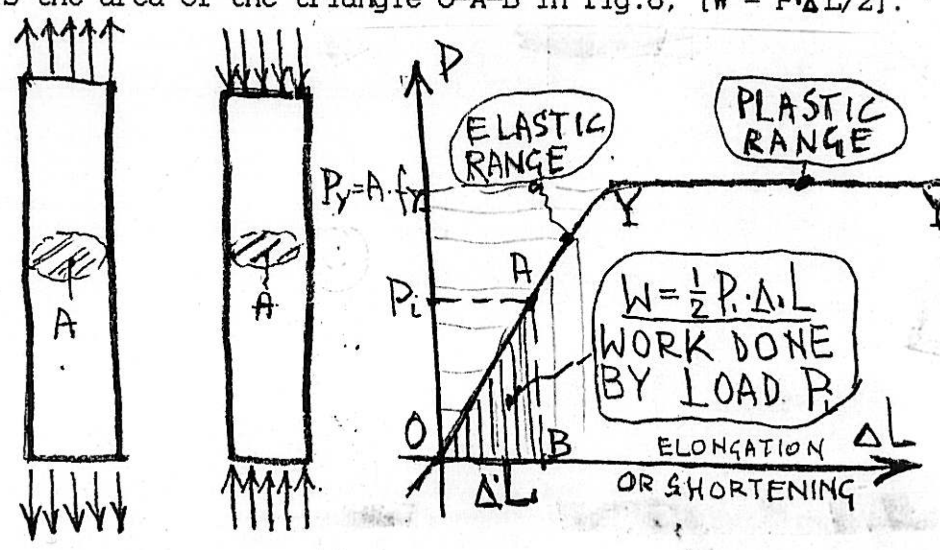
Work:

$$W=\frac{1}{2}P\cdot\Delta L=\frac{1}{2}P\frac{P}{AE}L=P^2\cdot L/2AE \qquad [1]$$

Elastic energy:

$$En=\frac{1}{2}f\frac{f}{E}L\cdot A=V\cdot f^2/2AE \qquad [2]$$

Work = Elastic energy [3]



8

E X A M P L E 1

Vertical column of constant section (its area A), firmly embedded into a massive ceiling and floor, is loaded in its quarter point by force P (figs 9). When the distance between floor and ceiling is assumed to remain unchanged, then load P becomes partially suspended from above and partially supported from underneath. In effect, the upper portion of the column is tensioned by the unknown force F_1 and the lower one compressed by F_2.

It is also assumed that the column behaves identically in tension as well as in compression.

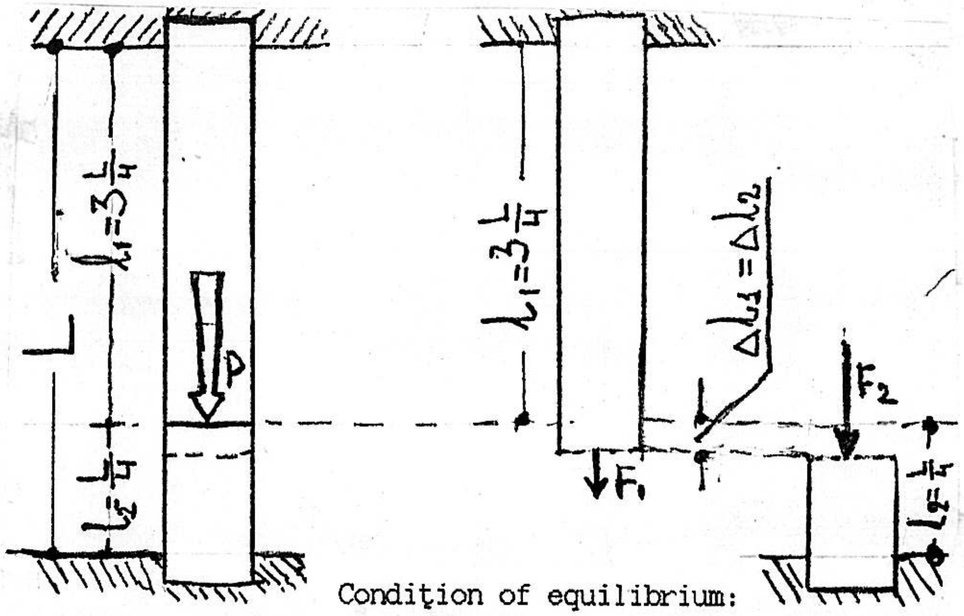


Fig 9.

Condition of equilibrium: $F_1 + F_2 = P$ [a]

The above equation [a] cannot yield (specific) values of F_1 or F_2, which we are looking for, because the case in consideration is "statically indeterminable", what means that additionally, deformations of both parts of the column ought to be taken into account. This is done by the equation of compatibility [b] saying that the elongation of the upper part of column equals the shortening of the lower part.

Since $\Delta l_1 = \Delta l_2$ then: $F_1 \cdot l_1/AE = F_2 \cdot l_2/AE$; and finally $3F_1 L/4AE = F_2 L/4AE$ [b]

Before even trying to calculate values of F_1 and F_2, one can predict that compressive force (F_2) in the shorter lower part has to be three times larger that tension (F_1) in the upper - three times longer -part, to produce numerically the same deformation of both. Such prediction is confirmed by solving jointly equations [a] and [b].

$F_1 = 0.25$ P (tension); $F_2 = 0.75$ P (compression)

Total elastic energy En in the column equals the sum of work done by both forces F_1 and F_2:

$$En = W_1 + W_2 = \tfrac{1}{2}F_1 \cdot \Delta l_1 + \tfrac{1}{2}F_2 \cdot \Delta l_2 = (0.75 \cdot F_1^2 + 0.25 \cdot F_2^2)L/2AE = 0.09375\ P^2L/AE. \quad [c]$$

To prove that the action of load P pulling the upper portion of the column with force $F_1 = 0.25P$ and pushing the lower one with $F_2 = 0.75$ P, results in the minimum elastic energy event, let's calculate the amounts of energy which would result from other sets of fictitious values of F_1 and F_2, still satisfying conditions of equilibrium [a], but being in obvious conflict with the geometrical requirement of compability of deformations [b].

	F_1	F_2	Elastic Energy
	0.00	P	0.12500 *
True, min.energy solution	0.25P	0.75P	0.09375 *
	0.50P	0.50P	0.12500 *
	0.75P	0.25P	0.21875 *
	P	0.00	0.37500 *

* $\dfrac{P \cdot L}{A \cdot E}$ [d]

The above comparison [d] confirms an earlier statement, that the elastic structure distributes its internal forces in a manner minimizing the total elastic energy involved.

Until magnitude of P is such, that stresses in the lower column do not exceed the maximum value of elastic stress (known as yield stress f_y), both parts of the column behave elastically. It means that their deformations remain proportional to forces carried by them. The segment o -1 in the diagram on fig 1o corresponds to the purely elastic behavior of the whole column.

When the lower column is fully stressed to value f_y, magnitudes of load P' and forces in column are:

$$F_1 = \frac{1}{3} A \cdot f_y \; ; \; F_2 = A \cdot f_y \; ; \; P' = \frac{4}{3} A \cdot f_y .$$

Further increase of load, beyond value P', is possible without endangering the integrity of the structure if its material is ductile (elasto-plastic). In the plastic stage such material does not break, but deforms (yields, flows) under constant force $f_y A$ – – an idealized plastic behavior of material is presented by horizontal line Y – Y in the simplified diagram P –Δ L (fig.8). Therefore, when the load acting on the column exceeds value $P' = 4/3\ A \cdot f_y$, its upper part (not yet fully stressed) assumes, with its reserves of resistance, the task of balancing any increment of load above value P', until this part of the column also reaches the limit of its opposition to tension. This means when F_1 equals $A \cdot f_y$.

The maximum (final, limit) strength of column P'' equals then $2 \cdot A \cdot f_y$, and any increase of load beyond this value will theoretically (and very probably – practically) cause the failure of the column. Both stages (elastic and elasto-plastic) of the response of the column to load are illustrated by the simple diagram on fig.1o.

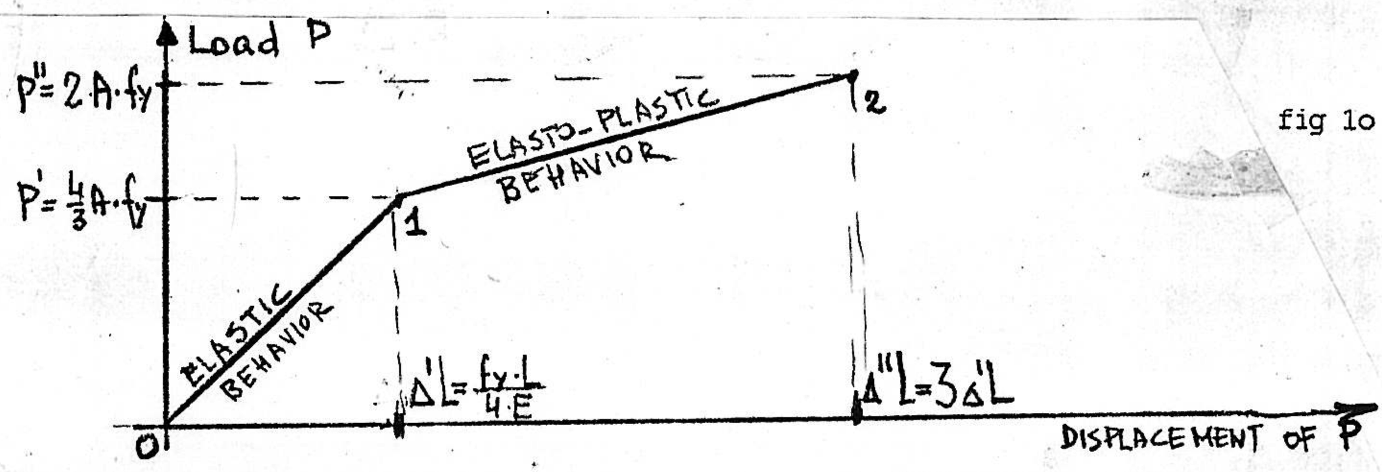
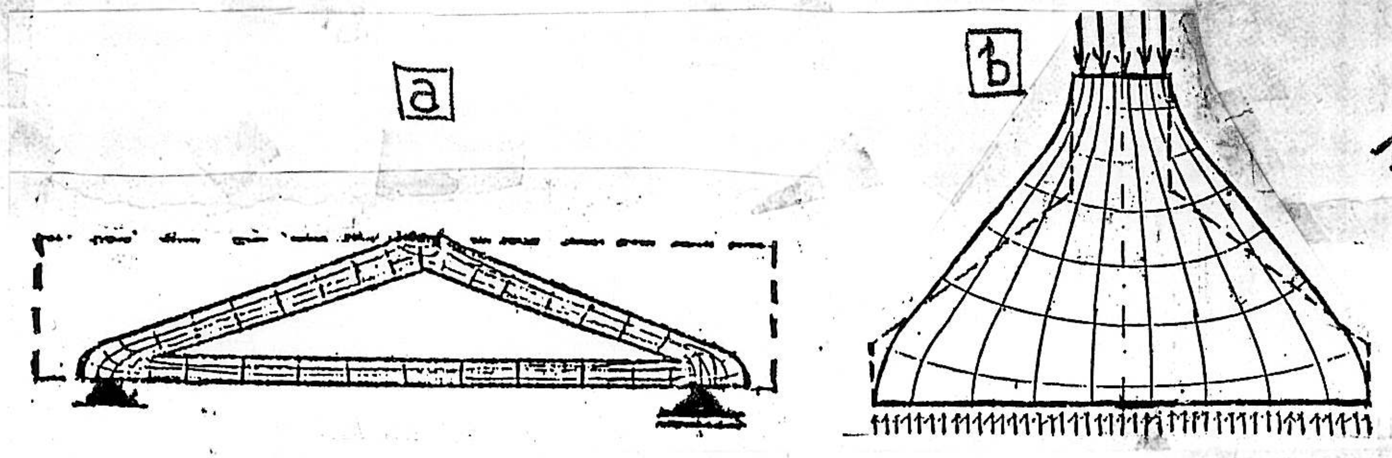


fig 1o

The just reviewed stages of response of a statically indeterminate column to load, may serve as the illustration to the following, more general description of behavior of all statically indeterminate structures;

DURING AN ELASTIC STAGE OF LOAD RESISTING ACTION, THE FLOW OF FORCES IN STRUCTURE ALWAYS PASSES THROUGH THE MOST FIRM (THE LEAST DEFORMABLE) EQUILIBRIUM ROUTE AND WHEN MATERIAL THERE STARTS TO YIELD, THE STRUCTURE FINDS OTHER STIFFER (SAFER) EQUILIBRIUM PATH IN A MANNER DELAYING FAILURE AS MUCH AS POSSIBLE.

Comparison of structures presented in figs 5, with their modified versions (identically supported and loaded) figs 11, reveals the influence of the shape of structure on the character of the flow of forces. Through these modifications a significant change in the flow of forces has been accomplished: e.g. the complex beam action (fig5a) has been substituted by the simple interaction of two inclined compressed members with one horizontal tie-member (fig 11a) –also the more direct (than in footing shown in fig.5b) penetration of forces into the modified shape is shown in figs.11b.



11o

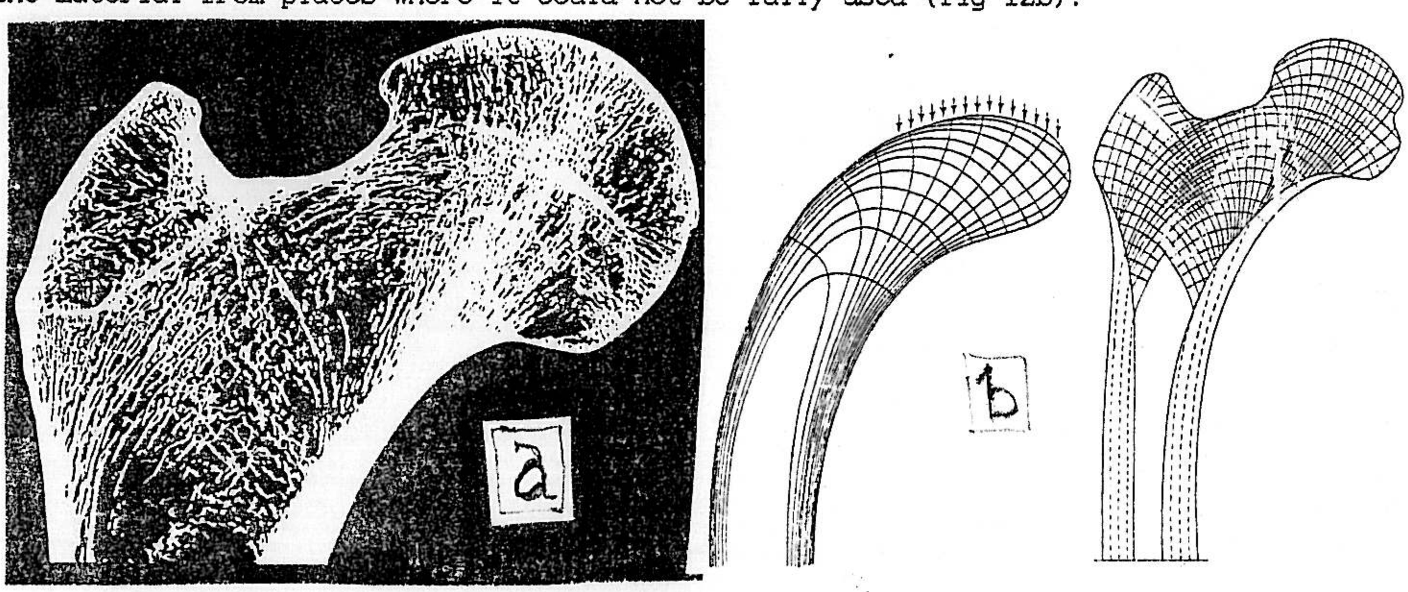
Numerous structural configurations exist or can be invented to resist the specific actions of gravity, winds or earthquakes. To facilitate the orientation among the vast number of possibilities, and to find the best (the most suitable, the lightest, the least expensive, the most attractive, etc) structure, the process called optimization should be employed.

"..;INSTEAD OF ANALYZING STRUCTURES TO DETERMINE STRESSES, WE SHOULD BE TREATING THE SUBJECT FROM THE VIEWPOINT OF FORCE TRANSMISSION" F.R. Shanley

An active designer, by creating or selecting an appropriate structural form, is imposing on his structure the desired predetermined kind of the flow of forces. If his design goal is the achievement of efficiency through the better use of properties of materials and the reduction of the structure's own weight, he is in good company of the Master Designer -Nature -practicing optimization as a strategy of survival.

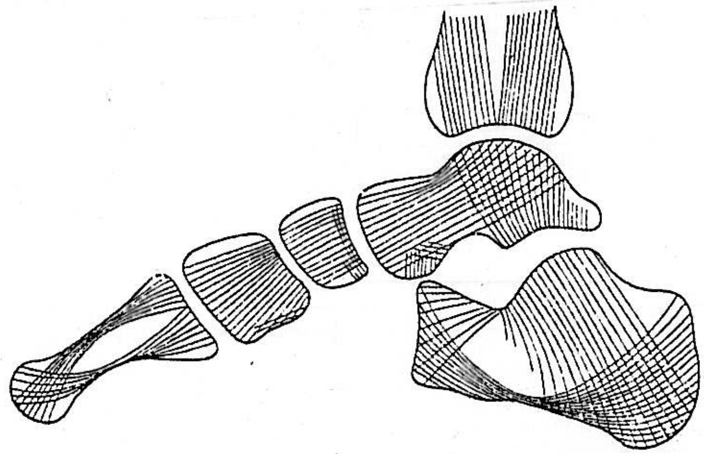
In biological structures, one of the remarkable instances of transfer of internal forces occurs in bones of most mammals, where parts of the bones' interior are filled with a fine lattice-work of cancellous tissue (fig 12a).

D'Arcy Thompson relates how in 1866 Karl Culman, one of the precursors of Graphical Statics, happened to visit an anatomist sectioning a bone of a shape similar to the structure of the huge crane which Culman was designing. In an instant, he realized that the arrangement of bone's cancellous tissue was just like a diagram of trajectories of principal stresses in his structure. He is said to have cried out, "that's my crane," having seen that Nature was strengthening the bone in precisely the manner and direction in which strength was required and eliminating the material from places where it could not be fully used (fig 12b).



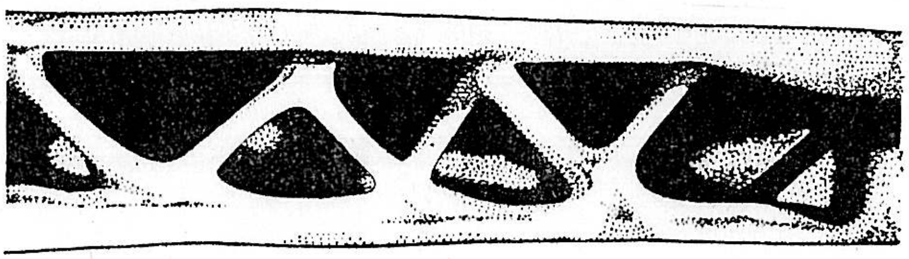
The same phenomenon of the adjustment of bone structure to the specific action of loads may be observed in many other cases. For instance; the orientation of the cancellous framework in the bones of the human foot has a definite relation to our basic erect position (13a). In all the mechanical side of anatomy nothing can be more beautiful - in the opinion of d'Arcy Thompson - than the construction of a bone from a vulture's wing (13b); "We see in it a real tri-dimensional truss".

a



figs 13

b



F O L L O W   T H E   P A T H   O F   F O R C E S

The act of creation, by nature, of structurally most rational forms, finds explanation in the fact that the growth of strong organic fibers and tissues is enhanced by their direct exposure to actions of internal forces. It is true not only for animal bones, but also for tissues in the trunk and branches of trees that become stronger according to forces which they have to carry -their cell fibers become aligned along trajectories of principal stresses, following directions of trunk and branches.

Since the proportions and dimensions of living structures tend to be misteriously optimized with regard to the strength they need for functioning and for survival, nature's own design and optimization methods are not available (at least not yet) to human designers obliged therefore to rely exclusively on mental and experimental speculations.

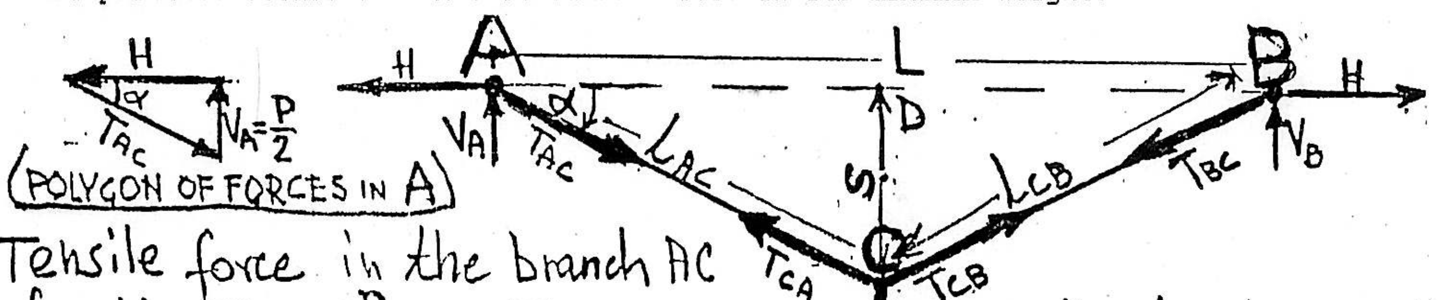
From among various empirical and analytical approaches utilized during the search for an ideal (optimal) structure, the study of routes frequented by internal forces seems to be one of the simplest and most productive.

Though the following elementary examples of structural optimization are limited to the weight of structural trusses versus their form relations, some important general conclusions can be drawn.

E X A M P L E 2

A) Flexible cable restrained at points A and B (distance between them = L) is loaded in the center by force P (fig.14a). The volume V of material in the cable with length l and such area (A) of cross section, as necessary to support load P with the allowable stress f_l is equal: length l times area A. Value of l decreases and of A increases with the reduction of sag s.

The problem at hand is to find the particular sag/span (s/L) ratio, resulting in the least possible volume V = A·l of cable - i.e. in its minimum weight.



Tensile force in the branch AC of cable; $T_{AC} = \frac{P}{2 \cdot \sin\alpha} = T_{BC}$

Length of each branch of cable; $l_{AC} = l_{BC} = \frac{L}{2\cos\alpha}$

The area of cable cross section A needed to resist force T_{AC} with the allowable stress f_{all}, equals: $\frac{P}{f \cdot 2\sin\alpha}$

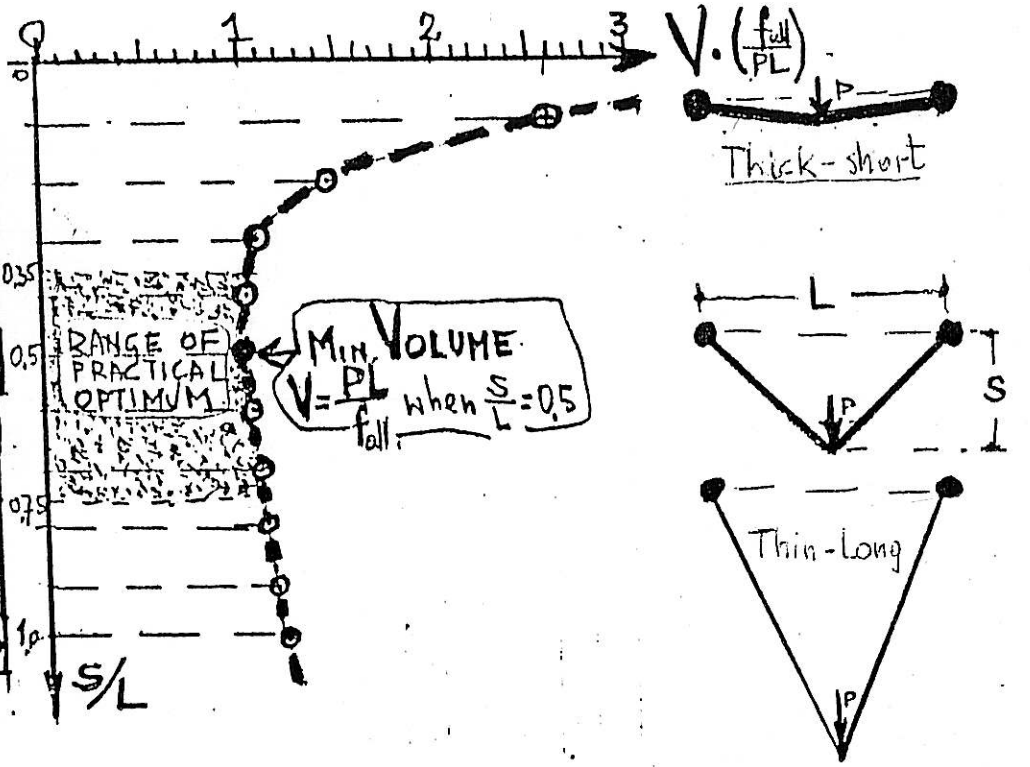
14a

The volume of material in both branches of cable;

$$V = 2 \cdot l_{AB} \cdot A = 2 \frac{L}{2 \cdot \cos\alpha} \cdot \frac{P}{f \cdot 2\sin\alpha} = \frac{PL}{f \sin 2\alpha} = \frac{P \cdot L}{4 f}\left[\frac{L}{s} + 4\frac{s}{L}\right] \quad [a]$$

V is minimum when $\sin 2\alpha = 1$; it means $\alpha_{MIN} = 45°$; $V_{MIN} = \frac{PL}{f}$ [b]

S	$V \cdot \frac{f_{all}}{PL}$
0	∞
0,1L	2,600
0,2L	1,450
0,3L	1,133
0,4L	1,025
0,5L	1,000
0,6L	1,017
0,7L	1,057
0,8L	1,113
0,9L	1,178
L	1,250



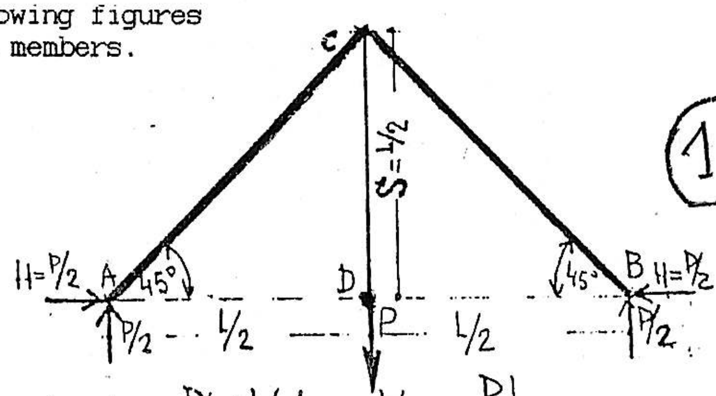
14b

The boldly dashed line in fig 14b, showing relations [a] between variable sag-span ratio (s/L) and the cable volume V (multiplied by f/PL), indicates that even 5o% deviations from the optimum ratio s/L=o.5 have only negligeable effect on the minimum value: V = P L/f.

Therefore, the whole range of values of s/L between o.35 and o.75 can be considered as corresponding to the minimum volume configuration of the fully stressed cable loaded in the center.

B) When load P is acting at point D (the center of line connecting supports of structure shown in fig14c) the volumes of material in all three elements have to be taken into account:

Note; Heavy lines will be used in the following figures to emphasize positions of compressed members.



14c

<u>Elements AC and CB:</u>

Length: $l_{AC} = l_{CB} = L/\sqrt{2}$

Compression: $C_{AC} = C_{CB} = P/\sqrt{2}$

Cross Section: $A_{AC} = A_{CB} = \frac{P}{f \cdot \sqrt{2}}$

Volume: $V_{AC} = V_{CB} = A_{AC} \cdot l_{AC} = \frac{PL}{2f}$

<u>Element CD</u>: Length $l_{CD} = L/2$; Cross section $A_{CD} = P/f$; Volume; $V_{CD} = \frac{PL}{2f}$

<u>Total volume</u>: $V = V_{AC} + V_{CB} + V_{CD} = \frac{P \cdot L}{2f} + \frac{P \cdot L}{2f} + \frac{P \cdot L}{2f} = 1.5 \frac{P \cdot L}{f}$

C) Determine now the volume of the structure in fig 14d, under load P in the center, suspended from the main framee by means of two inclined hangers. (fc and ft are values of allowable compressive and tensile stresses in all elements).

Element AC: $l_{AC}=\frac{L}{2}$; $C_{AC}=\frac{P}{\sqrt{2}}=0{,}707P$

$A_{AC}=\frac{P}{f^c\sqrt{2}}$; $V_{AC}=\frac{PL}{f^c\cdot 2\cdot\sqrt{2}}=0{,}3535\frac{PL}{f^c}$

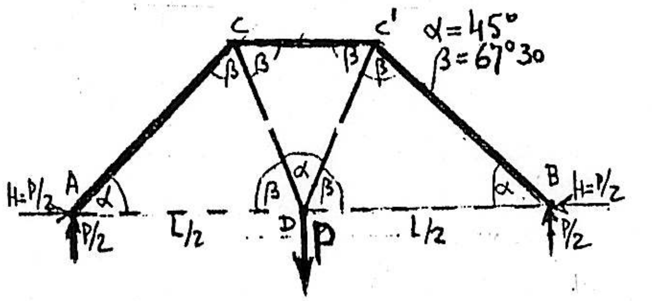
Element CD: $l_{CD}=2\frac{L}{2}\sin 22°30'=0{,}3827L$.

$T_{CD}=\frac{P}{2\cdot\cos 22°30'}=0{,}5412P$

$V_{CD}=\frac{1}{f^t}0{,}5412P\times 0{,}3827L=0{,}207\frac{PL}{f^t}$

Element CC': $l_{CC'}=L-\frac{L}{\sqrt{2}}=0{,}293L$; $C_{CC'}=0{,}707P$

$V_{CC'}=\frac{1}{f^c}0{,}707P\cdot 0{,}293L=0{,}207\frac{PL}{f^c}$



14d

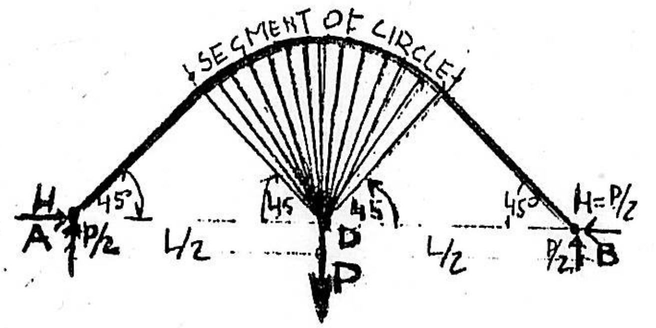
TOTAL VOLUME $V=2V_{AC}+V_{CC'}+2V_{CD}$

$$V=2\cdot 0{,}3535\frac{PL}{f^c}+0{,}207\frac{PL}{f^c}+2\cdot 0{,}207\frac{PL}{f^t}=PL\left[\frac{0{,}914}{f^c}+\frac{0{,}414}{f^t}\right]; \text{ When } f^c=f^t,\ V=1{,}328\frac{P\cdot L}{f}$$

D) Fig 14e shows framework similarly loaded as that on fig 14d, but lighter.

$V_{MIN}=PL\left[\frac{0{,}8925}{f^c}+\frac{0{,}3925}{f^t}\right]$

When $f^t=f^c$; $V_{MIN}=1{,}285\frac{PL}{f}$



14e

E X A M P L E 3

The addition of the tie-members A-B to frameworks shown in figs 14d and 14e transforms them into free supported trusses (figs 15a,b).

$V=PL\left[\frac{0{,}914}{f^c}+\frac{0{,}414}{f^t}\right]+\frac{PL}{2f^t}=0{,}914PL\left[\frac{1}{f^c}+\frac{1}{f^t}\right]$

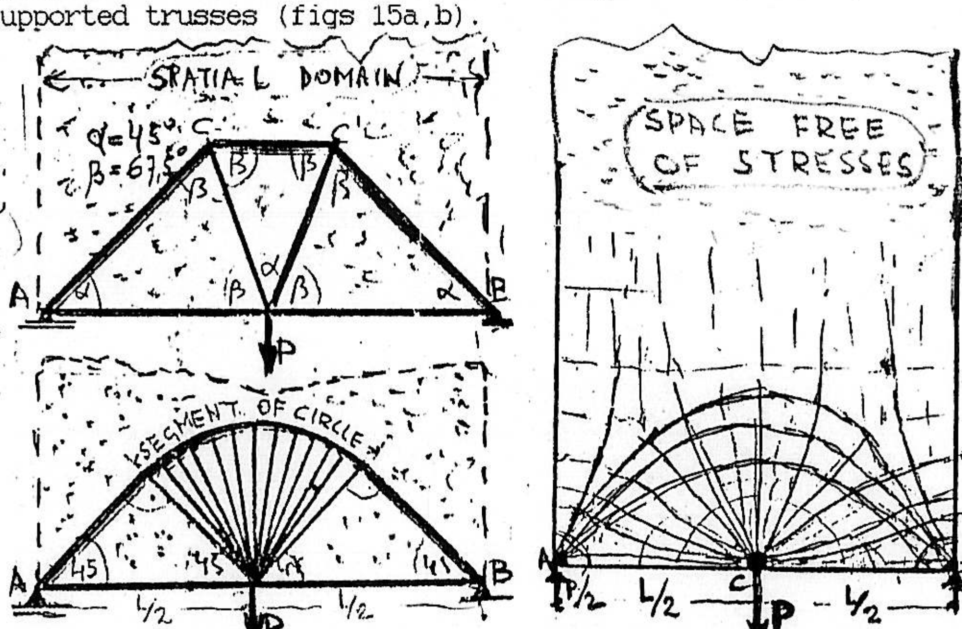
When $f^c=f^t$; $V=1{,}828\frac{PL}{f}$

a

b

$V=PL\left[\frac{0{,}8925}{f^c}+\frac{0{,}3925}{f^t}\right]+\frac{0{,}5PL}{f^t}=0{,}8925PL\left[\frac{1}{f^c}+\frac{1}{f^t}\right]$

$f^c=f^t$; $V=1{,}785\frac{PL}{f}$



15

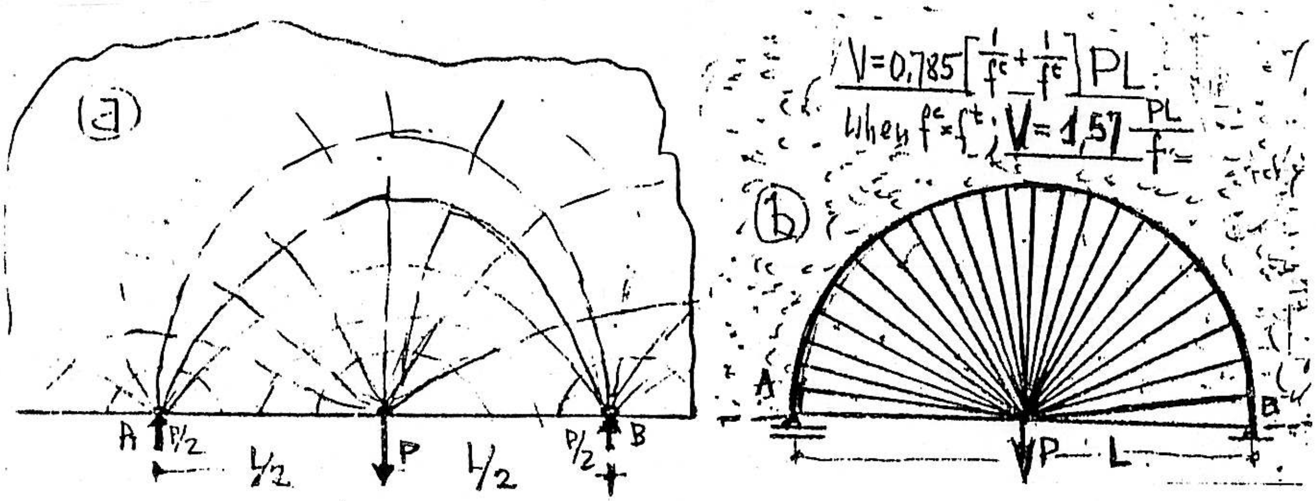
Fig.15a shows the practical and fig.15b, the theoretical minimum weight configuration of trusses that can be built in the space above line of supports and between two vertical lines (dotted) passing through A and B.

Fig 15c depicts the outline of trajectories of principal stresses in an infinitely high, solid slab of constant thickness, with the same boundary as the space available for the development of similarly loaded trusses (15a,b). Notice the resemblance of the pattern of lines of internal forces in the wall, to the geometry of both optimal trusses.

Trusses lighter than those just shown can be found when the spatial domain available for their development is increased.

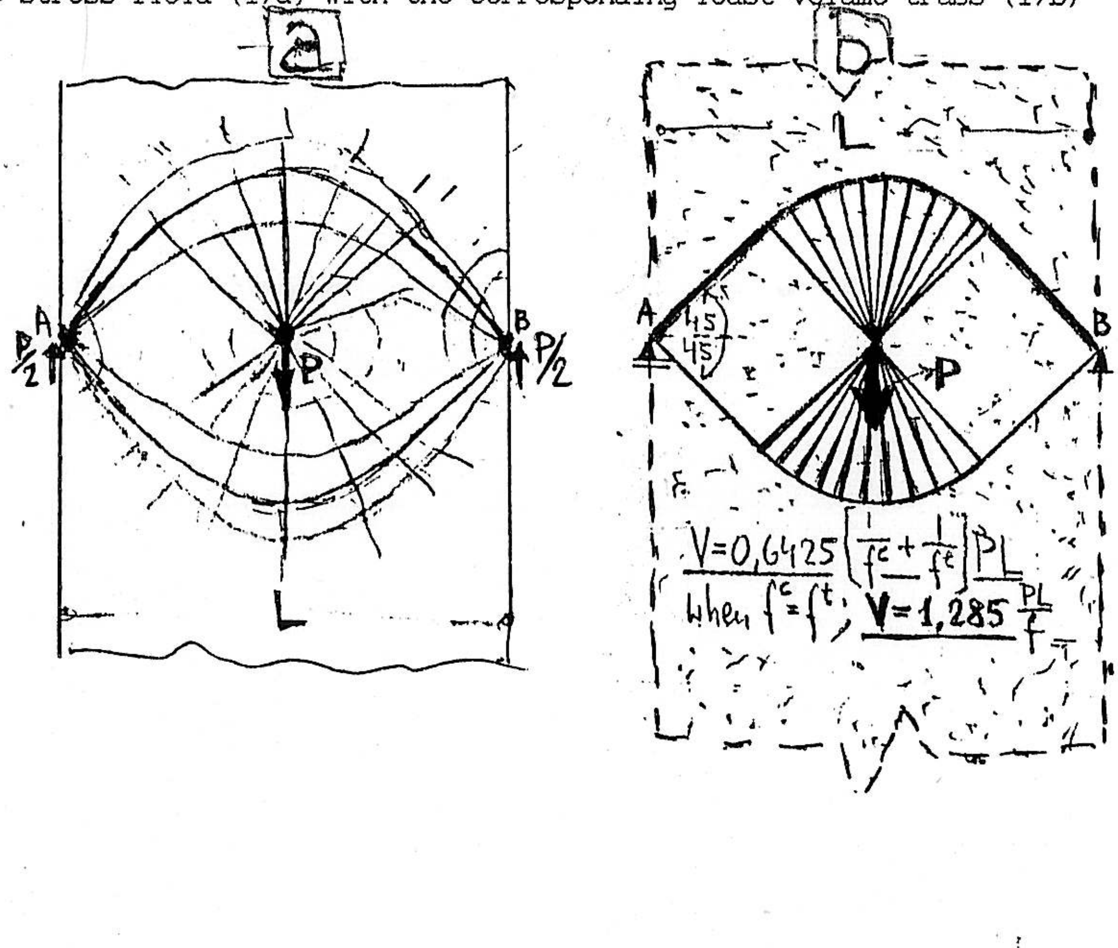
EXAMPLE 4

A) Lines of forces (stress field) in the wall extending infinitely above line A-B (a half-plane space), shown in fig 16a, can be compared with the form of the minimum-weight framework developed in the same space - fig 16b.



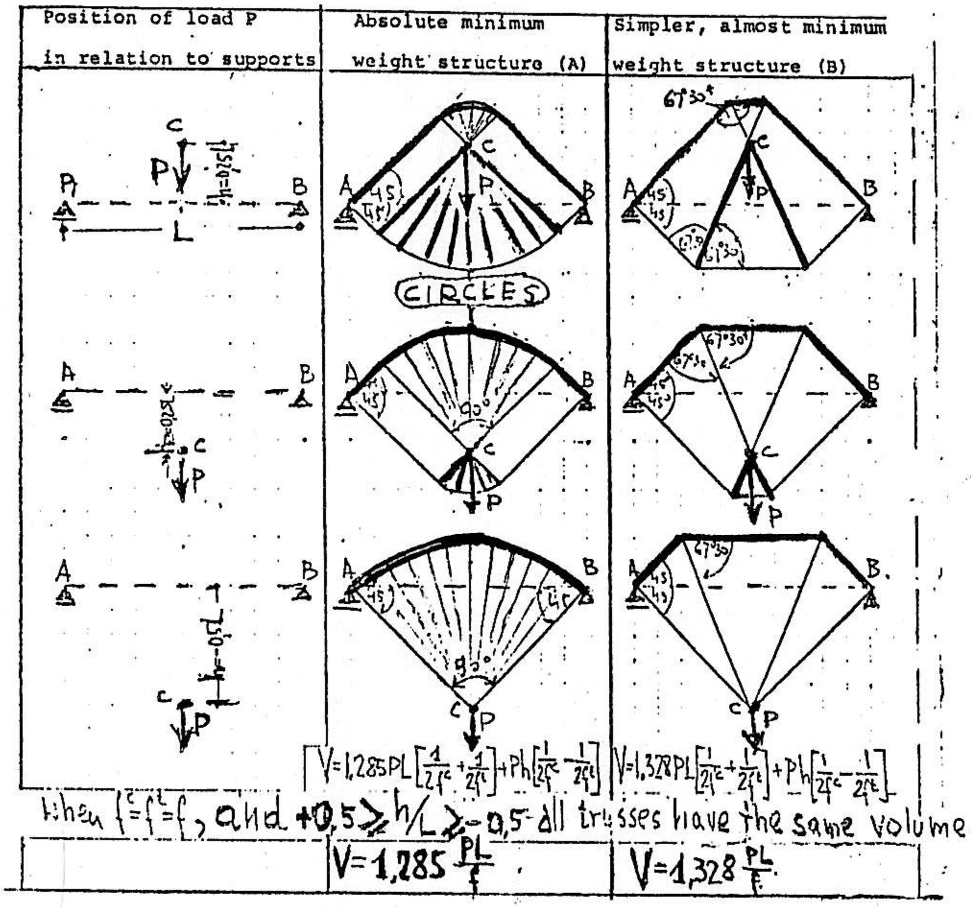
16

B) Space in the narrow strip of slab contained between vertical lines passing through points A and B (supports of trusses) offers more (above and beneath line A-B) opportunities for the development of an absolutely minimum weight truss under given load and support conditions.
Compare the stress field (17a) with the corresponding least volume truss (17b)



17

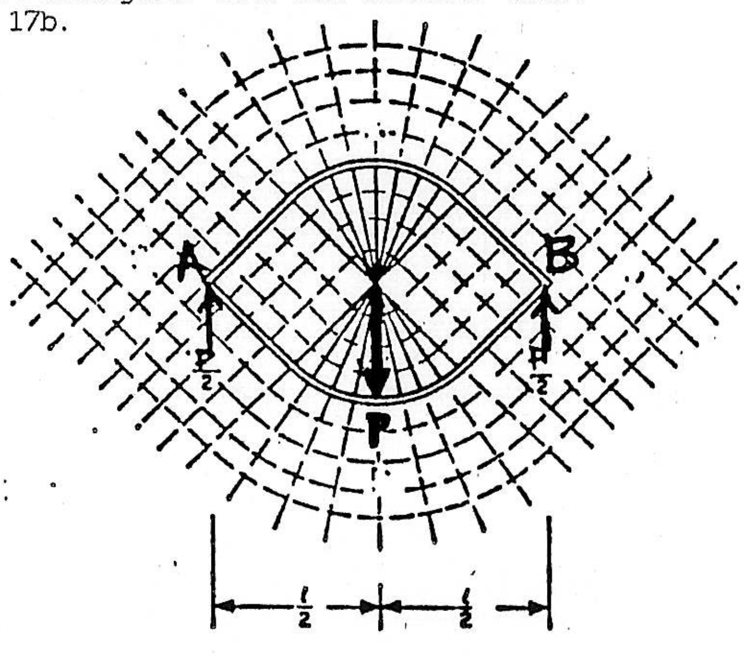
C) Figs 18 show two versions; one theoretical and another simplified, of minimum-weight frameworks under middle load P, applied in points above and beneath the line of support A-B.



18

The optimal trusses presented until now, are known as "Michell Structures" after A.G.M. Michell, who has deduced and in 1904 announced important criteria that the minimum volume framework must satisfy; its members must meet perpendicularly at each joint or form circular fans with members radiating from points of loading and reactions.

Fig.19 shows one of Michell's classical frameworks on the background of a the orthogonal grids made of rectangular nets and circular fans.
Compare with figs 7 and 17b.



19.

In all members of the presented least volume frameworks, stresses are assumed to be always equal to the allowable values of tension f_t and compression f_c (effects of buckling were not accounted for). Therefore, the total elastic energy stored in these structures, can be expressed by eq [4], in terms of the sum (V = Vc + Vt) of volumes of tensioned and compressed members, by extending over the whole structure the meaning of an earlier expression [2].

$$En = Vc \cdot fc^2/2E + Vt \cdot ft^2/2E; \quad \text{when } fc = ft = f \text{ then } En = V^{Tot} \cdot f^2/2E \qquad [4]$$

Thus, since - in such a situation - the stored energy is proportional to the volume of structure, the lightest one absorbs less energy than any other, functioning in the same spatial domain and resisting the same loads with the same stresses.

Because energy stored in any structure is equal to the work done by loads acting on it - a further important property of least-volume structures can be derived, regarding their deflections Yc under mid-span load Pc - eq.[5a].

$$\text{Since work } W = \frac{1}{2} Pc \cdot Yc = En = V^{Tot} f^2/2E \text{ - then } Yc = 2 \cdot En/Pc = V \cdot f^2/EPc \qquad [5a]$$

Note, that volumes of all frameworks examined until now (with the same stress in all members) can be presented in a general form given by eq.[5b]:

$$V = \Omega\, P \cdot L/f \qquad [5b]$$

In [5b],the dimensionless coefficient Ω depends (for a given load and support conditions) on the form of structure, hence [5a] can be transformed into [5c].

$$Yc = \Omega L \cdot f/E \qquad [5c]$$

According to eqs. [5a] and [5c],the least-volume (minimum weight) framework is , in a general sense, also the least deformable (the most rigid) of all possible frameworks supporting the same load with the same stresses.

Example: deflection of the absolutely least-weight framework known (Ω= 1.285) (fig.17b) becomes:

$$Y = 1.285 \cdot L \cdot f/E$$

When mild steel (E = 29ooo Ksi) is used and stresses everywhere are 23 ksi, the mid-point displacement of the above framework equals:

$$Y = 1.285 \cdot L \cdot 23/29000 = L/1000$$

CONCLUSION:

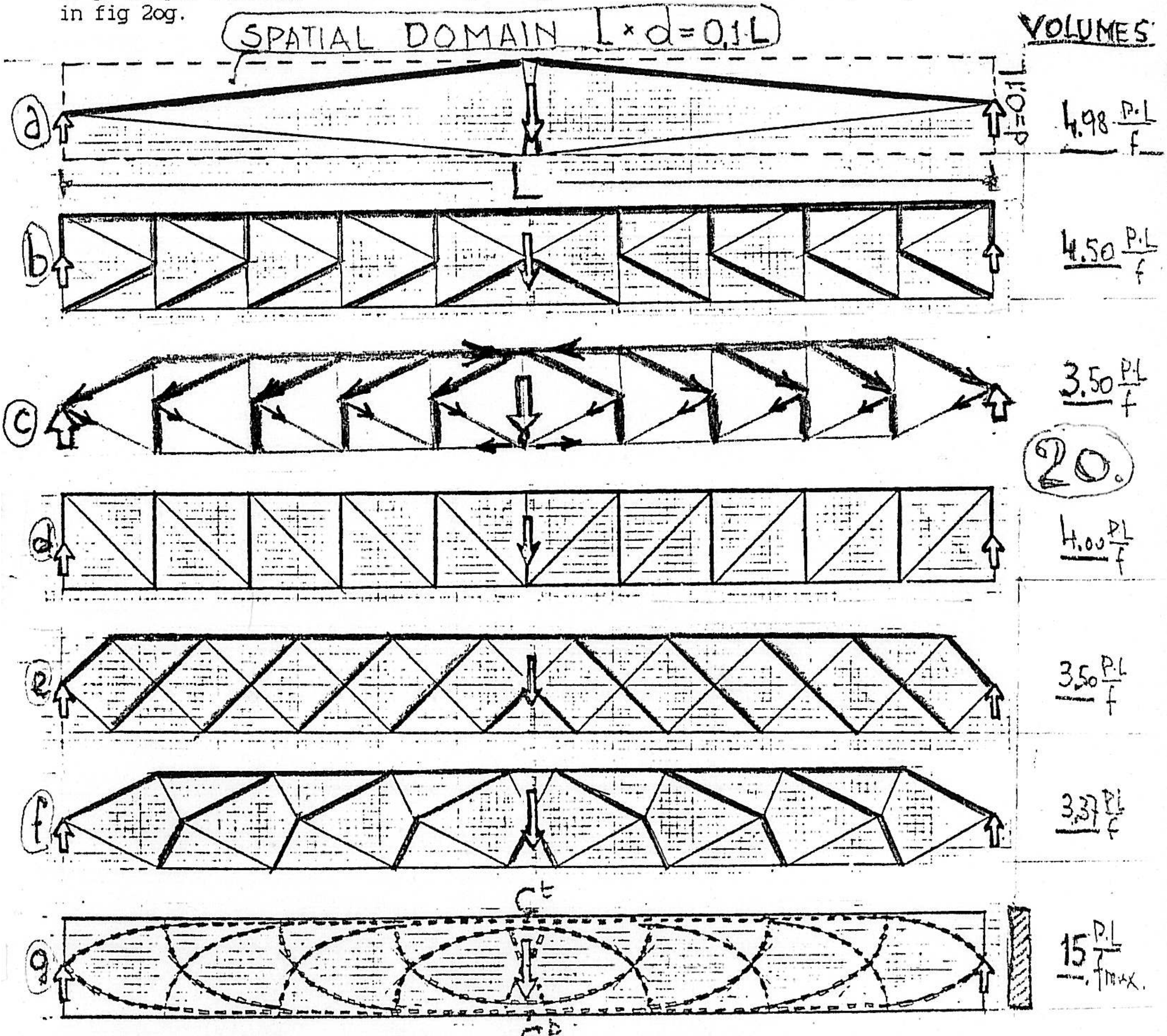
ANY REAL FREE SUPPORTED STEEL STRUCTURE, LOADED IN THE CENTER, WILL DEFLECT <u>ALWAYS</u> MORE THAN THE MOST RIGID ONE SHOWN IN FIG (17b), OR MORE THAN 1/1ooo L (ONE THOUSAND OF SPAN) - IF STRESSES EVERYWHERE ARE AT LEAST 23 Ksi.

The presentation of some elementary ideas of structural optimization based on the geometrical and physical deductions, has been limited until now, to a few cases illustrating the existence of the close similarity (though not identity) between outlines of the flow of forces or "stress fields" in solid bodies and forms of optimum frameworks developed in the same spatial domain and subjected to the same action of forces. This similarity is of a general character and is not limited to those amazing but hardly practical forms shown until now.

EXAMPLE 5

Comparison of six different bar arrangements in trusses (developed in the rectangular spatial domain L times L/1o - figs 2o a,b,c,d,e and f) loaded in the center, gives the least volume award to truss "f" (V = 3.375 P·L/f).

Not surprisingly, its geometrical order approximates best the outline of trajectories of principal stresses in the corresponding solid beam of similar depth-span ratio, shown in fig 2og.



In the solid beam with rectangular cross-section (2og), max. value of stress is reached only in points Ct and Cb.

The prevailing pattern of stress trajectories in beams is of a lattice-work type (fig 7c), producing a clearly recognizable arch-like arrangement of compressed members interacting with the analogous, but reverse, system made of members in tension.

The image of interior "arches" supporting one part of the load on beams and "cables" supporting the rest of it is so natural, that the "arch -cable" bon mot became an almost standard figurative description of the characteristic modes of the flow of forces in beams.

To prove the corectness of the "arch-cable" image of behavior of prismatic beams, from their efficiency point of vue, let's compare volumes of two so-called K trusses (figs 2ob and c) which though very similar, have opposite orientations of diagonal members. Truss (c) exhibits clearly developed arched routes of compression and suspended routes of tension whereas the form of the truss (b) ignores the good example of nature.

Results: the "unnatural" truss (b) uses almost 3o% more material to support (with the same stresses) the same load as truss (c) shaped more "naturally"- truss (b) also deflects 3o% more.

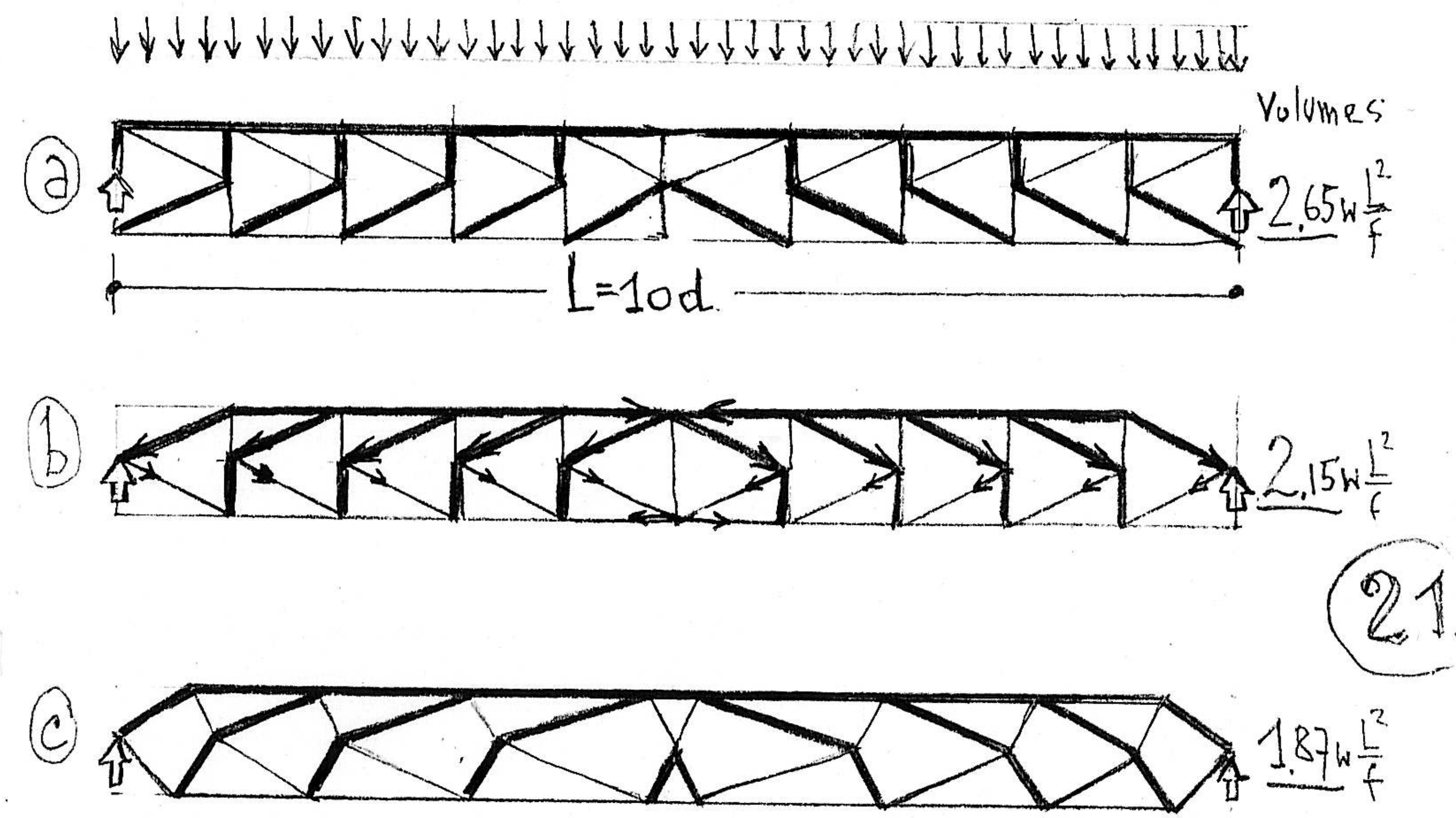
Since the arch-cable pattern of stress trajectories in beams develops under any type of loading (and not only under action of a single load in the center) an assumption can be made that, the low-weight forms of truss-beams in almost all cases may be developed on the same principle as used until now.

If the geometry of such trusses were extracted more formalistically, with less intuitive insight and less geometric reasoning, the abstract and much more laborious, than in case of a single load, process of derivation could become inaccessible to all those who are not well versed in mathematical operations.

To examine the validity of the assumption, that the "stress field - optimal shape" affinity exists independently on the type of loading, three uniformly loaded beam-trusses are mutually compared in figs 21.

Once more the "follow the forces" precept proved to be a very good advice, which has been taking well by truss "b" and almost literally by truss "c" -with excellent results. Truss "a" however, which did not ask for advice became punished with more than 4o% of over-weight (cost) and a much reduced rigidity.

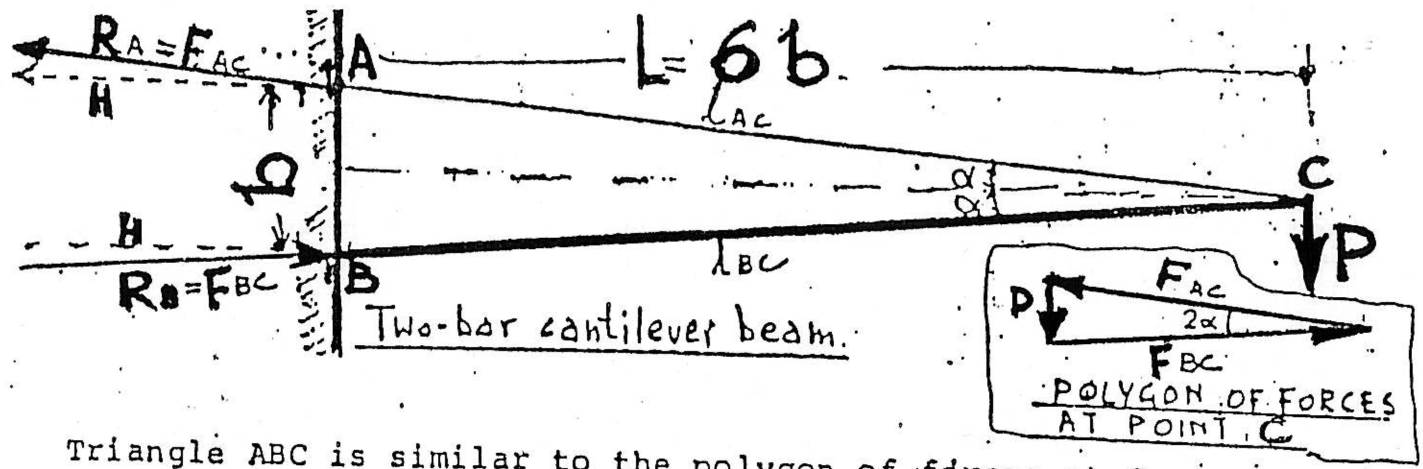
UNIFORMLY DISTRIBUTED LOAD w [weight per unit of span]



In calculations of forces in members of above trusses and of their volumes, the distributed load has been substituted by an equivalent system of forces acting in joints on the neutral axis of each truss.

EXAMPLE 6

a) Determine the total volume of a two-bar cantilever structure, fig 22a.



Triangle ABC is similar to the polygon of forces at C.

$\frac{l_{AC}}{b} = \frac{F_{AC}}{P}$; $F_{AC} = P\cdot\frac{l_{AC}}{b}$; Volume $V_{AC} = \frac{F_{AC}}{f^t}\, l_{AC} = P\frac{l^2_{AC}}{b\cdot f^t}$

$\frac{l_{BC}}{b} = \frac{F_{BC}}{P}$; $_{BC} = P\cdot\frac{l_{BC}}{b}$; Volume $V_{BC} = \frac{F_{BC}}{f^c}\cdot l_{BC} = P\frac{l^2_{BC}}{b\cdot f^c}$

$l^2_{AC} = l^2_{BC} = (b/2)^2 + (6b)^2 = 36{,}25b^2$

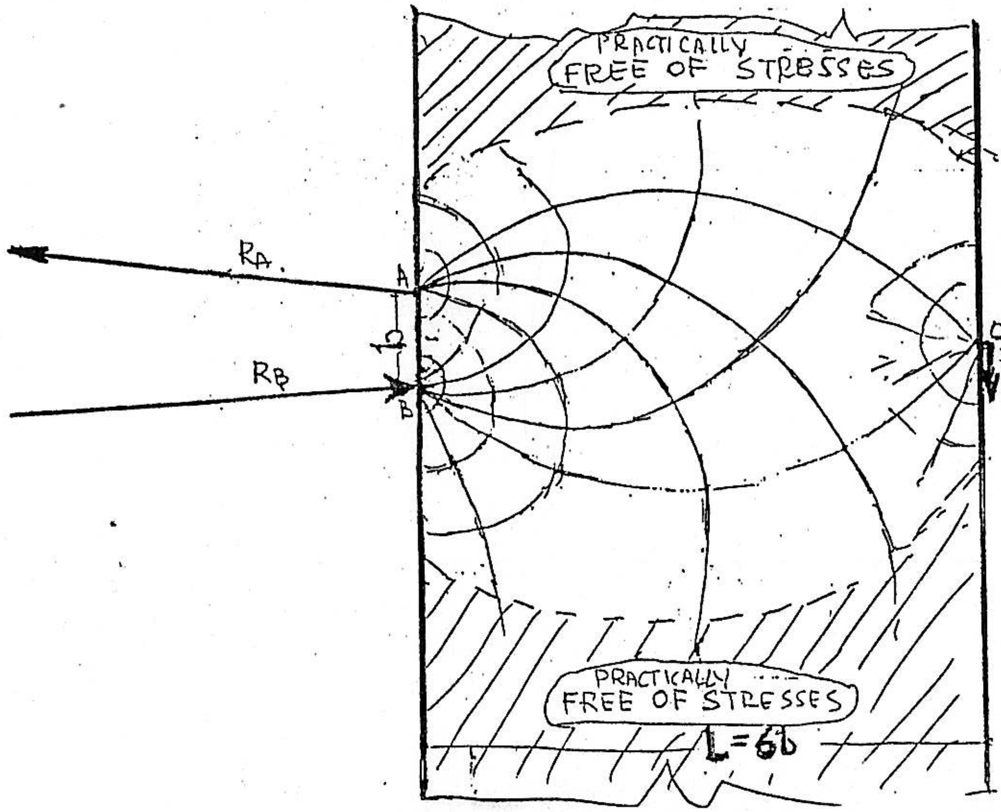
The total Volume $V = \frac{P\cdot l^2_{AC}}{b}\left[\frac{1}{f^t} + \frac{1}{f^c}\right] = 36{,}25\left[\frac{1}{f^t} + \frac{1}{f^c}\right]\cdot b$ 22a

When $f^t = f^c$; $V = 2\frac{bP\cdot 36{,}25}{f} = 72{,}5\frac{P\cdot b}{f} = \mathbf{12{,}1}\frac{P\cdot L}{f}$

The vertical displacement of point C by means of Form [5a]

$$Y_c = \frac{f^2}{E\cdot P}V = \frac{f^2}{E\cdot P}\,72{,}5\frac{P\cdot b}{f} = 72{,}5\frac{f}{E}\cdot b = 12.1\,\frac{f}{E}\cdot L$$

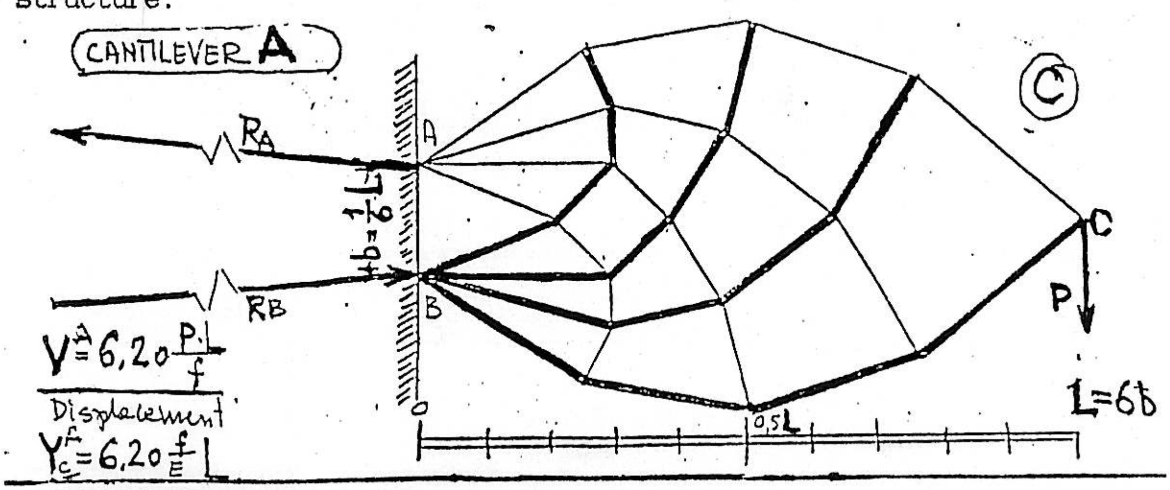
b) The approximate low-volume form of the cantilever truss not restricted in its depth and subjected to the same load as the two-bar structure (22a), can be deduced by following the example of the stress field in the corresponding vertical strip of wall (22b) with width L = an extension of the cantilever.
Forces acting on the wall are the same - load P (tangent to the edge of wall) and reactions R_A (pull) and R (push) -as those acting on the two-bar structure. The interference of two pushing and pulling fans (caused by reactions) combined with the third one (caused by P) results in another version of the lattice-work pattern of trajectories - fig 22b. Since forces acting on the wall are in equilibrium, then, according to the principle of St.Venant, the stress field in the wall develops only in the space around and between points where loads and reactions are applied, while the remaining parts of the wall are practically free of stresses.



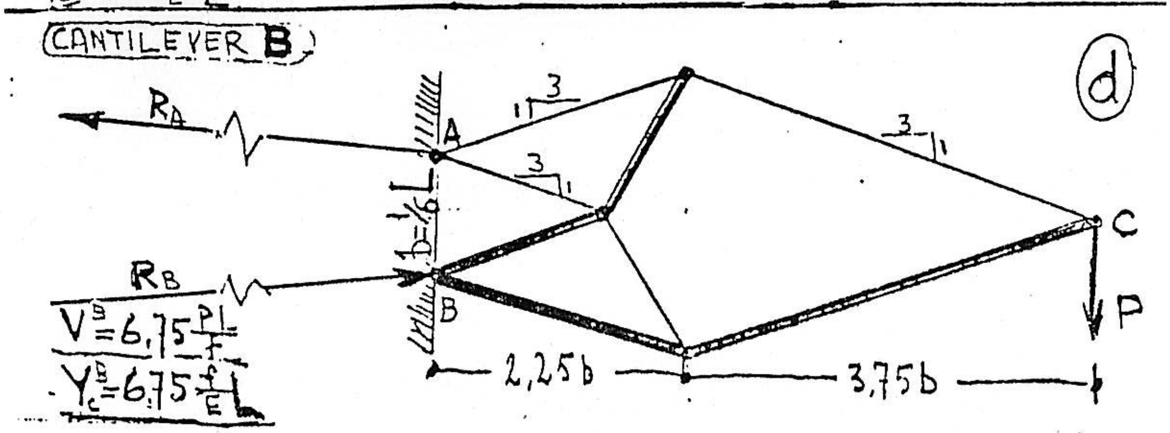
22b

Figs. 22c and 22d present two trusses whose shapes has been inspired by the outline of stress trajectories in the corresponding solid wall (22b). The volume of material in cantilever A (with the arrangement of bars closely approximating the stress field in the wall) equals: V = 6.2o P·L/f –about half of volume V = 12.1 P·L/f of the two-bar structure (fig 22a).

Another cantilever B, much simpler than A, but preserving the characteristic bulging shape of the corresponding stress field, has volume V = 6.75 P L/f –still much less than a two-bar structure.

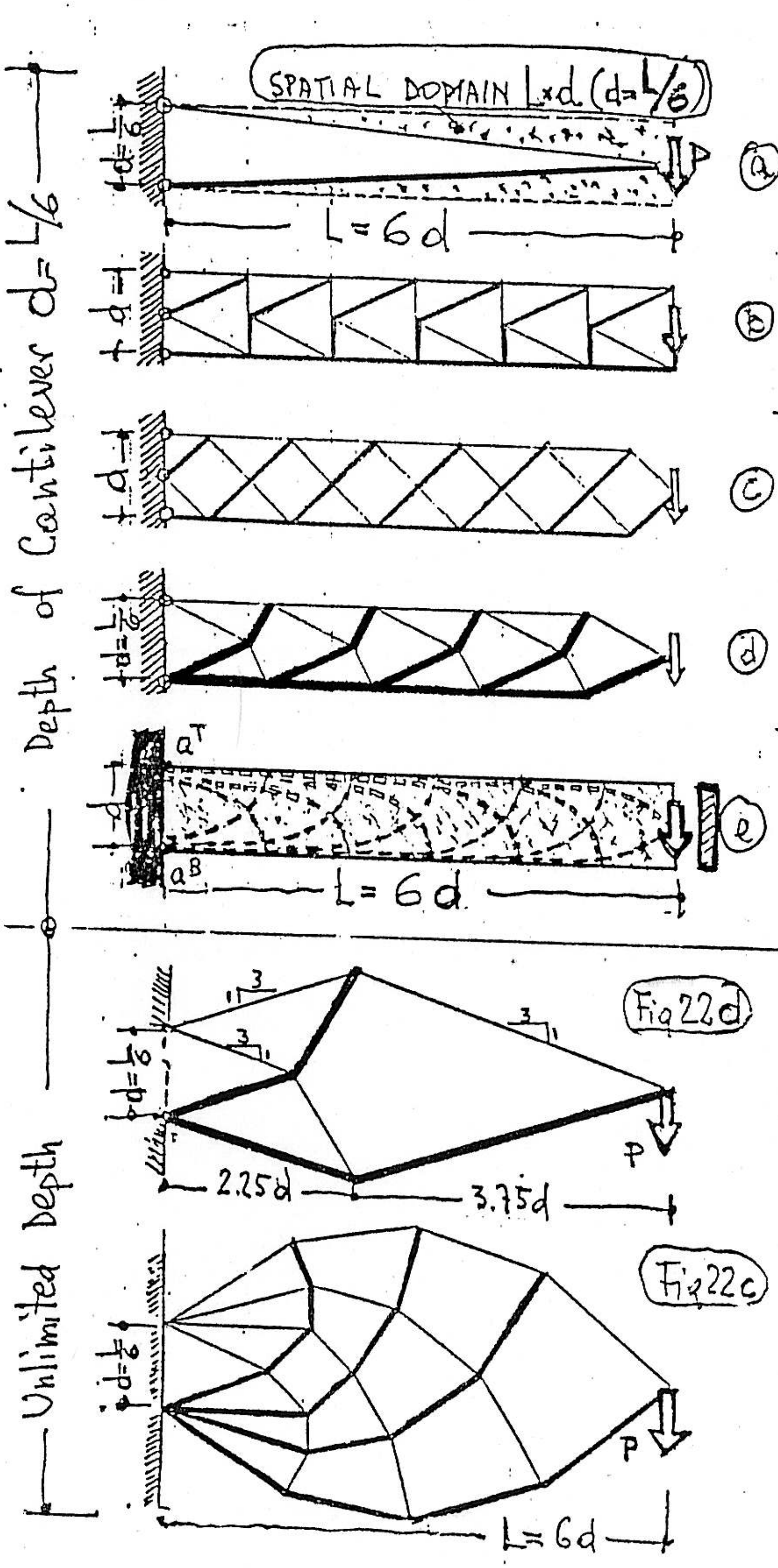


22c



22d

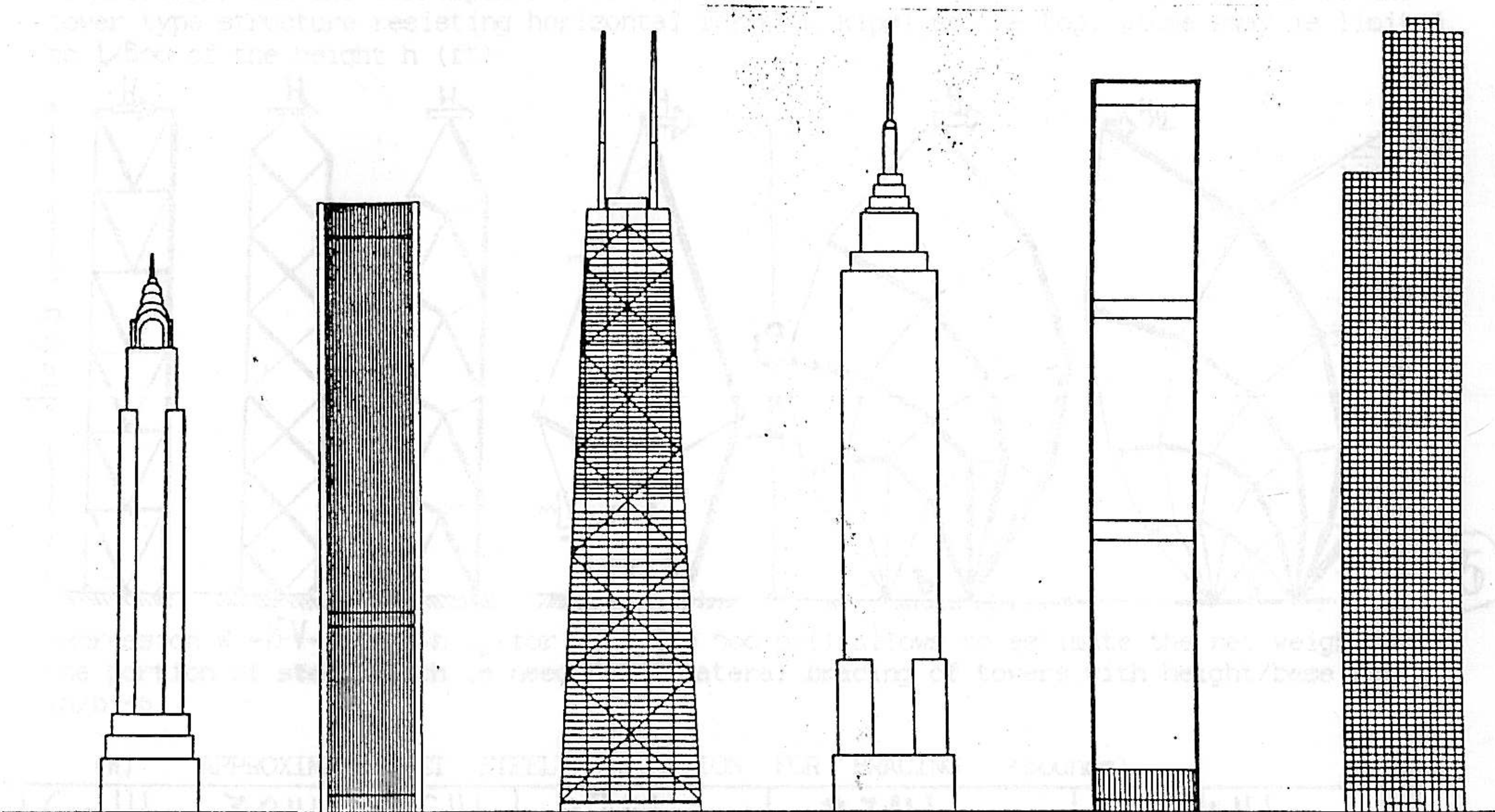
When the space available for the development of the trussed cantilevers is limited to a rectangle with a depth = 1/6 of L, the geometry of the stress field in the solid cantilever beam (fig 23e) with the same depth/extension ratio offers the best advice for a low weight design. Indeed, framework (23d) most similar to it, represents an almost optimum solution in a constrained space; it is compared with other trusses developed in the same (L, d = L/6) and in the vertically unconstrained (L, d = ∞) spaces.



	Volumes	Deflections.
	$V = \Omega \cdot \frac{P}{f} L$	$Y = \Omega \cdot \frac{f}{E} \cdot L$
(a)	$V_a = 12.08 \frac{P \cdot L}{f}$	$Y_a = 12.08 \frac{f}{E} L$
(b)	$V_b = 10.00 \frac{P \cdot L}{f}$	$Y_b = 10.0 \frac{f}{E} L$
(c)	$V_c = 8.00 \frac{P \cdot L}{f}$	$Y_c = 8.00 \frac{f}{E} L$
(d)	$V_d = 7.71 \frac{P \cdot L}{f}$	$Y_d = 7.71 \frac{f}{E} L$
(e)	$V_e = 36 \frac{P \cdot L}{f_{max}}$ IN THE SOLID CANTILEVER - f_{max} IS REACHED ONLY IN POINTS a^T and a^B	$Y_e = 4.0 \frac{f_{max}}{E} L$
	$V_B = 6.75 \frac{P \cdot L}{f}$	$Y_B = 6.75 \frac{f}{E} L$
	$V_A = 6.20 \frac{P \cdot L}{f}$	$Y_A = 6.20 \frac{f}{E} L$

23

Let's now examine the possibility of using some of the just presented shapes of cantilever trusses as frameworks of tower buildings, which structurally are huge vertical cantilevers subjected (besides gravity loads) to the lateral, often violent actions of wind and earthquakes - fig 24.



The design of high-rise structures is usually influenced not only by the requirement of their adequate resistance to all probable actions, but also by the necessity to restrain the lateral sways, which can be too uncomfortable for the users. This requirement is met, during design, by the imposition of a limit on the horizontal displacements of the building top - usually allowed to be no more than 1/5oo of the height.

The demand for a sufficient structural firmness puts restrictions on values of f_H - this portion of total stresses ($f_{TOT} = f_V + f_H$) which is caused by horizontal actions of wind and earthquakes; f_V corresponds to vertical loads. By introducing stress f_H into expression [6a]-based on the general formula [5c] - we can roughly estimate the sway Y of tops of buildings (height = h) stiffened by the structure in which the average increase of stresses caused by lateral (horizontal) actions equals f_H.

$$Y = \Omega\, h \cdot f_H / E. \qquad [6a]$$

Thus, the requirement of limitation of lateral movements of high-rise structures can be expressed by eq [6b]:

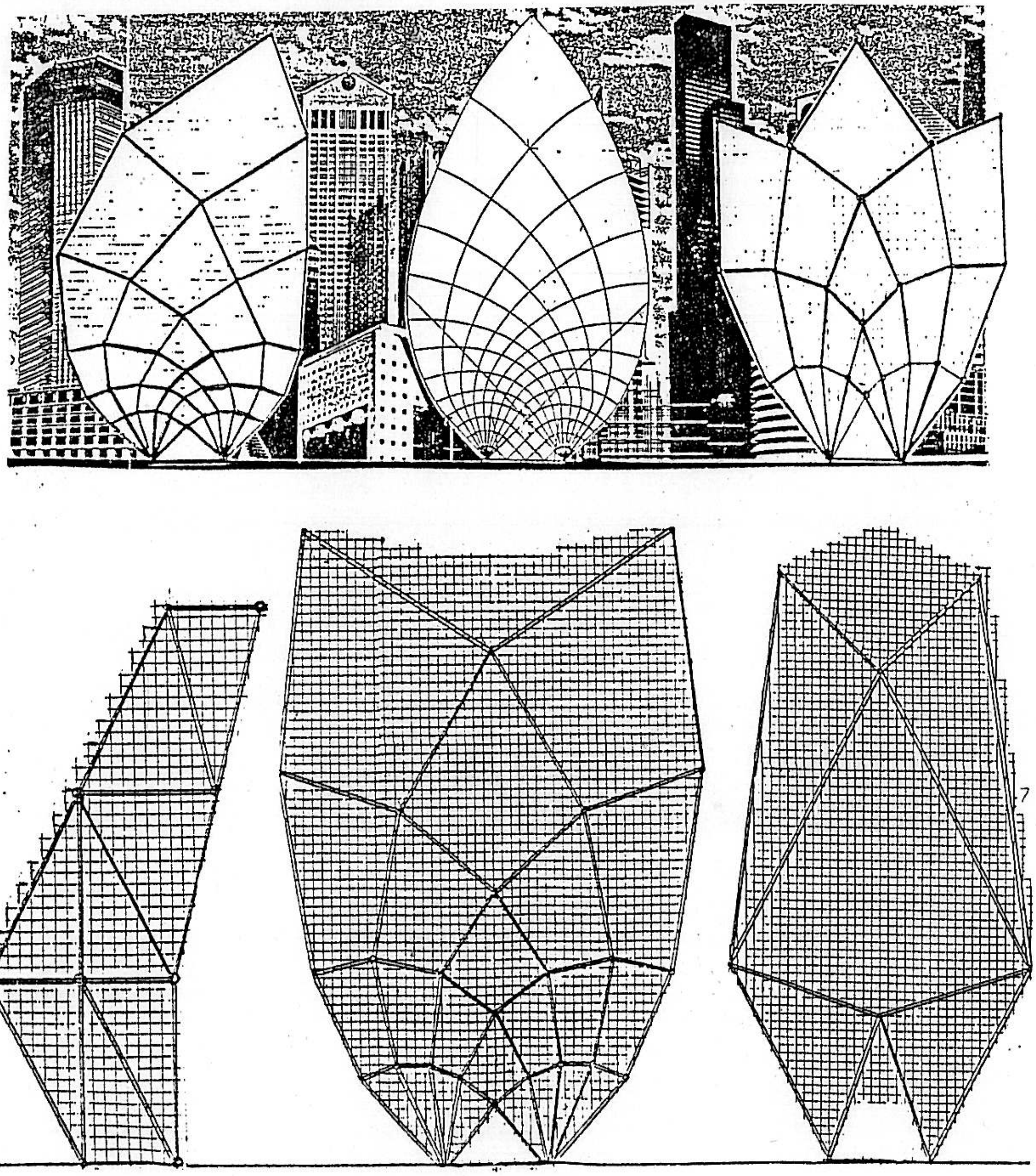
$$Y/h = \Omega\, f_H / E \leqslant 1/5oo \qquad [6b]$$

As a result, the average, limit value of stress f depending solely on horizontal actions and related to the desired limitation of the sways of a building can be approximately determined by means of eq.[6c]:

$$f_H \leqslant E / 500\,\Omega \qquad [6c]$$

The origins of the imagination stirring forms shown in figs 27 with their clear (architectural and economic) potential can be traced directly to Michell mathematical creations.

However, only the semi-intuitive approach used in the phenomenological interpretation of the principle of energy conservation makes it broadly accessible even to non-specialists and creates the conceptual environment conducive to the new ideas - some of them could make structural sense.



27

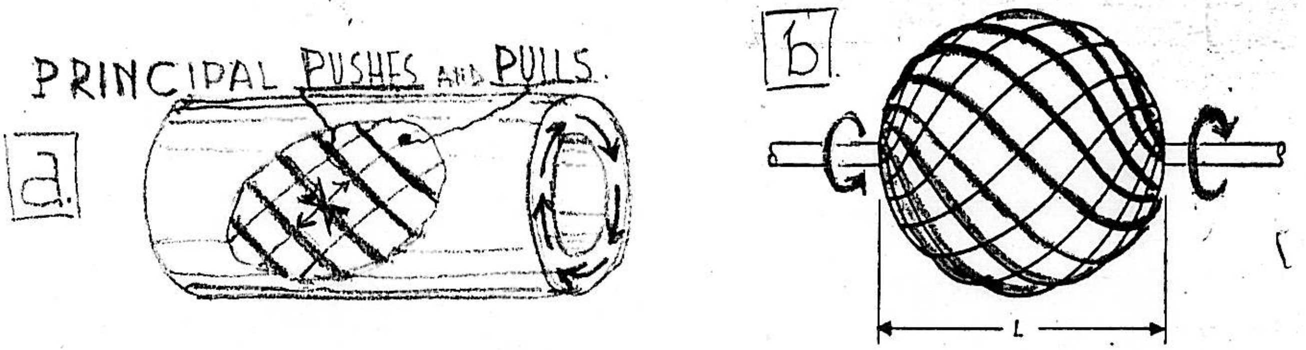
One of the essential points of the present considerations comes from the recognition of the existance of pictorial affinity between forms of optimum frameworks and the patterns of stress fields in equivalent solid bodies. Such affinity is the visually explicit manifestation of the firm control exerted in both cases by the Nature`s law of energy conservation - what allows the designer to use the same three basic orthogonal nets satisfying Michell's criteria of least-volume frameworks, as a geometrical background for practically all possible modes of transfer of internal forces.

"THE USUAL PROCEDURE IN STRUCTURAL ANALYSIS INVOLVES THE DETERMINATION OF THE STRENGTH OF A CERTAIN STRUCTURE UNDER SPECIFIC LOADING CONDITIONS. FROM THE DESIGN POINT OF VIEW THE REAL PROBLEM IS TO DETERMINE THE LIGHTEST PRACTICAL ARRANGEMENT OF MATERIAL THAT WILL TRANSMIT THE REQUIRED LOADS THROUGH THE SPECIFIED DISTANCES" F.R. Shanley

The "design with Nature" rule is of a broad general character, serving well also in situations when the 3-directional interplay of internal forces has to be taken into account.

E X A M P L E 7

The hollow shaft (fig28a) is usually thought to be the most efficient shape for transmitting torque (torsion), though already in 19o4 Michell, in his celebrated paper, argued mathematically that a special spherical unit composed of a series of helically wound rods (fig 28b) would be stronger and lighter. Not until more than half a century later the impetus to apply this idea in practice has been given by the space age needs, where weight is anathema. Michell`s torque transmitting devices are now considered as feasible for certain earth-and space-bound equipment, because of their superior strength to weight ratio.

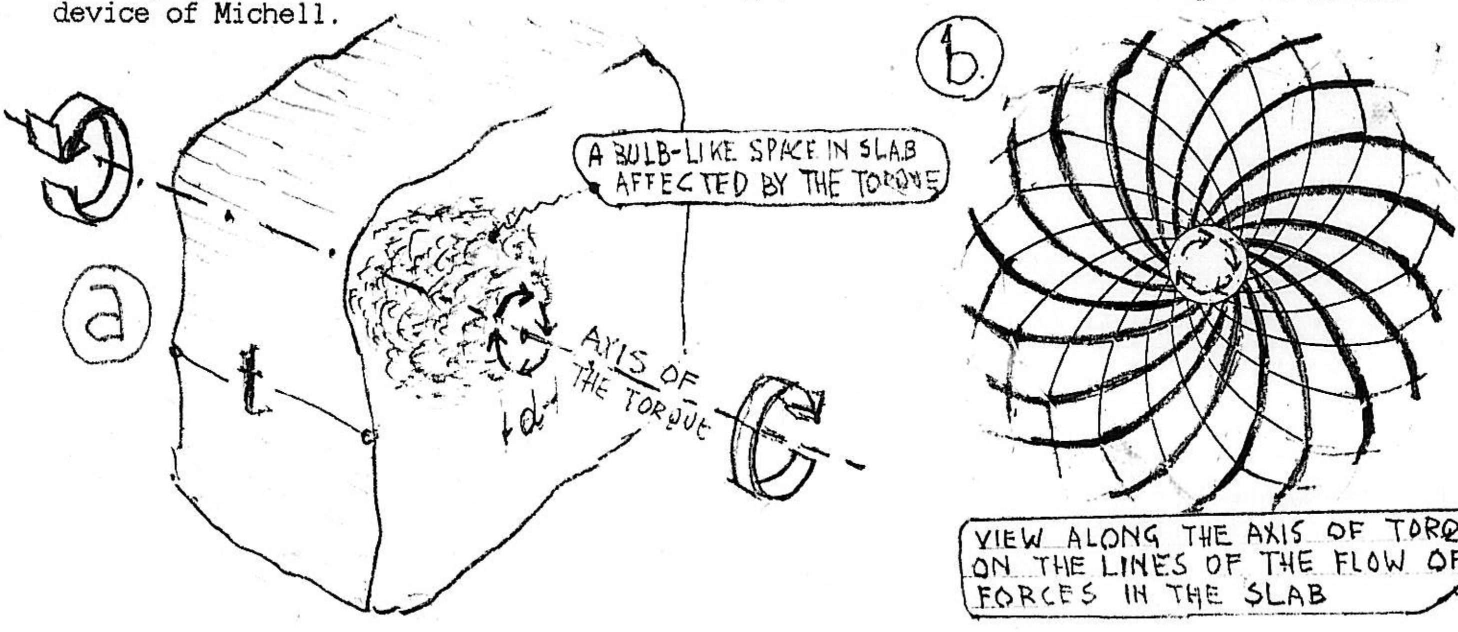


28

The shape of the least-weight torsional device will be now compared with the image of the flow of forces across a solid slab subjected to two opposing torques exerted on its exterior surfaces by circularly distributed tangential forces (the diameter d of circles of torques is much smaller than t the slab thickness), fig 29 a.

Internal forces transferring torsional action from one face of the slab to the other, are spreading through its bulk (but only in the vicinity of places of torque application) along helical spatial trajectories of tensions and compressions. Vectors of resultants of these forces form a bulbous surface whose view taken in the direction perpendicular to the slab is presented in (29 b).

It is no accident but the expression of the general rule, which we already know, that such spatial vectorial formation is strikingly similar to the least-weight torsional device of Michell.



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